Arizona State University, Physics 311
Problem Set #6, Rev. A, due Wednesday, 2014-03-05

1. Sagging tent potential. Solve Laplace's equation for the potential $V(x, y)$ in the region $-a<x<a$ and $-b<y<b$ with boundary conditions $V(x, \pm b) = 0$ and $V(\pm a, y) = V_0(1 - |y/b|)$. Sketch the solution.

2. The purpose of this exercise is to systematically develop the **multipole expansion** in cylindrical coorindates using a boundary value problem. We will solve Laplace's equation for the potential $V(r, \theta)$ inside and outside a spherical shell of radius r', due to a uniformly charged ring of total charge q at the polar angle θ' on the shell. In the limit $\theta' \to 0$, it becomes a point charge at the top of the sphere.

a) Write down a formula for the surface charge density $\sigma(\theta)$ of the ring charge distribution using a delta function of θ. Normalize it so that the total charge is $\int \sigma \, da = q$.

b) Repeat part a) in terms of the variable $x = \cos\theta$. Confirm the the change-of-variables formula for delta functions: $\delta(\theta - \theta')d\theta = \delta(x - x')dx$ where $x' = \cos\theta'$.

c) Solve $V(r, \theta)$ both inside and outside the shell due to the above ring charge distribution.

d) Calculate the potential in the limit $\theta' \to 0$, noting that $P_\ell(1) = 1$ always. Compare your answer with $V(r) = q/4\pi\epsilon_0 \mathscr{r}$ to derive the addition formula [Griffiths Eq. 3.94].

e) Substitute $q \to \int \text{đ}q'$ and factor out the resulting integral over the charge distribution to obtain the following multipole expansions for an azimuthally symmetric charge distribution:

$$V_{\text{ext}}(r, \theta) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} Q_{\text{int}}^{(\ell)} \frac{P_\ell(\cos\theta)}{r^{\ell+1}} \qquad \text{where} \qquad Q_{\text{int}}^{(\ell)} = \int \text{đ}q' r'^{\ell} P_\ell(\cos\theta') \tag{1}$$

$$V_{\text{int}}(r, \theta) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} Q_{\text{ext}}^{(\ell)} \, r^\ell P_\ell(\cos\theta) \qquad \text{where} \qquad Q_{\text{ext}}^{(\ell)} = \int \text{đ}q' \frac{P_\ell(\cos\theta')}{r'^{\ell+1}} \tag{2}$$

Note that the external multipole potential is only valid outside the entire charge distribution, while the internal multipole potential is only valid completely inside the charge distribution; otherwise the potential will have a mixture with both contributions.

f) Compare this result with Griffiths Eqs. 3.95, 3.97, and 3.98–99. Identify the terms corresponding to the monopole, dipole, and quadrupole terms. What is potential inside an external monopole (uniform shell of charge)? What is the correspondence between the internal/external multipoles and the coefficients A_ℓ and B_ℓ of the general solution to Laplace's equation by separation of variables in cylindrical coordinates?

Also, Griffiths 3ed[4ed] Ch. 3, #13[14], 22[23], 24[25], 45[52]a,b.