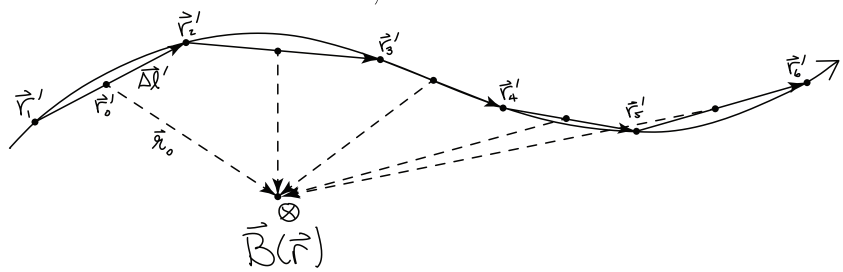
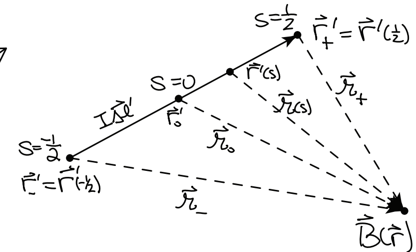
Arizona State University, Physics 311
Problem Set #8, Rev. A, due Wednesday, 2014-04-02

1. Thin wire—the Biot-Savart law can be integrated numerically by approximating a curved wire with a sequence of discrete straight current elements of small, but finite length:

$$\boldsymbol{B} \;=\; \frac{\mu_0}{4\pi}\oint' \frac{I\boldsymbol{dl}'\times\boldsymbol{\mathscr{r}}}{\mathscr{r}^3} \;\approx\; \sum_i \left(\Delta\boldsymbol{B}_i = \frac{\mu_0}{4\pi}\frac{I\Delta\boldsymbol{l}_i\times\boldsymbol{\mathscr{r}}_{0i}}{\mathscr{r}_{0i}^3}\right), \tag{1}$$

where $\Delta\boldsymbol{l}$ is the displacement vector from the beginning to the end of each current segment, and $\boldsymbol{\mathscr{r}}_0$ is the displacement vector from the middle of each current segment $\boldsymbol{r}_0'$ to the field point $\boldsymbol{r}$. The approximation is that all of the current is concentrated at $\boldsymbol{r}_0'$ instead of spread out along the length of the segment from $\boldsymbol{r}_0' - \Delta\boldsymbol{l}/2$ to $\boldsymbol{r}_0' + \Delta\boldsymbol{l}/2$. In this problem we first calculate a correction term to account for this difference, and then calculate the exact B-field due to each straight segment.





a) To analytically integrate the Biot-Savart law along a single straight segment of the path, parametrize the segment $\boldsymbol{r}'(s)$ with the parameter s, ranging from $s = -\frac{1}{2}$ at the beginning to $s = +\frac{1}{2}$ at the end of the segment. The parametrization should also include the constant vectors $\boldsymbol{r}_0'$ (the center of the segment) and $\Delta\boldsymbol{l}$ (displacement along the segment). Calculate the line element $\boldsymbol{dl} = \frac{dr'}{ds}ds$. Calculate $\boldsymbol{\mathscr{r}}$ as a function of $\boldsymbol{\mathscr{r}}_0$, $\Delta\boldsymbol{l}$, and s. Substitute these into the Biot-Savart formula and factor out the constant zeroth approximation of the field to obtain the form

$$\Delta\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu 0}{4\pi}\frac{I\Delta\boldsymbol{l}\times\boldsymbol{\mathscr{r}}_0}{\mathscr{r}_0^3}T(\alpha,\beta), \tag{2}$$

where the integral $T(\alpha,\beta)$ of s depends on the constants $\alpha = \boldsymbol{\mathscr{r}}_0\cdot\Delta\boldsymbol{l}/\mathscr{r}_0^2$ and $\beta = \Delta l^2/\mathscr{r}_0^2$.

b) Approximate the integrand of $T(\alpha,\beta)$ to order s^2 and integrate to obtain the correction term $T(\alpha,\beta)\approx 1+\frac{1}{8}(5\alpha^2-\beta)$ for the case where all the current is at the center of the segment.

c) Calculate the exact integral $T(\alpha,\beta)$ and show that

$$\Delta\boldsymbol{B} = \frac{\mu_0 I}{4\pi}\frac{\Delta\boldsymbol{l}\times\boldsymbol{\mathscr{r}}_0}{(\Delta\boldsymbol{l}\times\boldsymbol{\mathscr{r}}_0)^2}\,\Delta\boldsymbol{l}\cdot(\hat{\boldsymbol{\mathscr{r}}}_+ - \hat{\boldsymbol{\mathscr{r}}}_-), \tag{3}$$

where $\boldsymbol{\mathscr{r}}_\pm = \boldsymbol{r} - \boldsymbol{r}'(\pm\frac{1}{2})$ is the displacement vector from each end of the segment to the field point and $\hat{\boldsymbol{\mathscr{r}}}_\pm = \boldsymbol{\mathscr{r}}_\pm/\mathscr{r}_\pm$. Show that this agrees with Eq. (5.35) of Griffiths, and is equal to

$$\Delta\boldsymbol{B} = \frac{\mu_0 I}{4\pi}\frac{(\boldsymbol{\mathscr{r}}_-\times\boldsymbol{\mathscr{r}}_+)(\mathscr{r}_- + \mathscr{r}_+)}{\mathscr{r}_-\mathscr{r}_+\,(\mathscr{r}_-\mathscr{r}_+ + \boldsymbol{\mathscr{r}}_-\cdot\boldsymbol{\mathscr{r}}_+)} \tag{4}$$

[See "Compact expressions for the Biot-Savart fields of a filamentary segment", Eq. (9).]

Also, Griffiths 3ed[4ed] Ch. 5, #3[3], 6[6], 7[7], 9[9], 11[11], 12[13].