Arizona State University, Physics 311
Problem Set #9, Rev. A, due Wednesday, 2014-04-09

1. Magnetic scalar potential

a) Calculate the divergence and curl of both sides of the Biot-Savart law to obtain Ampère's law $\nabla \times \boldsymbol{H} = \boldsymbol{J}$, and $\nabla \cdot \boldsymbol{B} = 0$, where $\boldsymbol{B} = \mu_0 \boldsymbol{H}$. Note the mixed symmetry: $\boldsymbol{D}$, and $\boldsymbol{B}$ are fluxes and $\boldsymbol{E}$ and $\boldsymbol{H}$ are flows, while $\boldsymbol{E}$ and $\boldsymbol{B}$ are the force fields (derive from potential energy/momentum) and $\boldsymbol{D}$ and $\boldsymbol{H}$ are auxiliary 'source fields', with ρ and $\boldsymbol{J}$ as their sources. So the correspondence between $\boldsymbol{E}, \boldsymbol{D}$ and $\boldsymbol{B}, \boldsymbol{H}$ depends on your point of view.

b) Use a Gaussian surface and an Ampèrian loop to derive the corresponding boundary conditions for $\boldsymbol{B}$ and $\boldsymbol{H}$, respectively, and compare them to the electrostatic case.

c) Show that a magnetic scalar potential U is only defined current-free regions ($\boldsymbol{J} = \boldsymbol{0}$) by substituting the definition $\boldsymbol{H} = -\nabla U$ into Ampère's law. Note that U is related to the flow of $\boldsymbol{H}$, just as the electrostatic potential V is related to the flow of $\boldsymbol{E}$. What is analogous equation of $-\nabla^2 V = \rho/\epsilon$ for U?

d) There is a more subtle condition for U being well defined in a region $\mathcal{R}$ with holes, for example, a torus. Re-express Ampère's law in terms of the scalar potential U. What constraint is required for U to be conservative (no discontinuities) in such a region $\mathcal{R}$?

e) Express the boundary conditions of b) in terms of U, as we did for the electric potential V.

f) Sketch the equipotential surfaces of an infinitely long solenoid. Is it possible to match up U continuously along the entire boundary, so that $U_{\text{in}}(\boldsymbol{r}) = U_{\text{out}}(\boldsymbol{r})$? If not, what is the discontinuity?

2. A **cos-theta coil** is a wound longitudinally along a cylinder instead of azimuthally like a solenoid. We will calculate the windings of a uniform transverse field using the scalar potential U.

a) Given a cylinder of radius $s = a$ about the z-axis from $z = -b$ to $z = b$, and a constant magnetic field $\boldsymbol{H} = H_0 \hat{\boldsymbol{x}}$ inside, calculate the normal component H_n everywhere on the cylinder.

b) Use the flux boundary condition a) and Laplace's equation of Problem 1 to solve for the potential U outside of an infinitely long cylinder, $b = \infty$.

c) Use the flow boundary condition (from Ampère's law) to calculate the current density on the cylinder necessary to produce these fields. What is the direction and magnitude of the current density $\boldsymbol{K}_a(\phi)$ on the cylinder?

d) Superimpose two coaxial cylinders, one of radius a inside and another of radius A with current in the opposite direction. Can the currents be adjusted such that $\boldsymbol{H} = 0$ outside the larger cylinder? (a rhetorical question!) Calculate the current densities $\boldsymbol{K}_a(\phi)$ and $\boldsymbol{K}_A(\phi)$ on each cylinder such that $\boldsymbol{H}_{\text{out}} = \boldsymbol{0}$ and $\boldsymbol{H}_{\text{in}} = H_0\, \hat{\boldsymbol{x}}$ on the outside and inside the two cylinders, respectively.

e) Solve parts (b)–(d) as a boundary value problem for the case of two cylinders of finite length $-b < z < b$. Calculate and sketch the field lines of $\boldsymbol{H}_{\text{mid}}$ inside the two cylinders. Calculate and sketch the current on the end caps.

f) Calculate the magnetic flux Φ_B flowing in and out of the inner cylinder. We will use this result later to calculate the inductance of such a coil, a rare case which is exactly calculable.

Also, Griffiths 3ed[4ed] Ch. 5, #13[14], 14[15], 15[16], 18[19], 20[21], 23[24], 39[41], 46[47].