University of Kentucky, Physics 335
Homework #2, Rev. B, due Thursday, 2021-10-13

1-d. Gaussian Moments—In this homework we will learn the techniques for evaluating Gaussian intetegrals, and to use them to investigate the covariance of the joint Gaussian integral.

a) We saw in H01 that a 2-d Gaussian was much easier to generate than 1-d because of its natural Jacobian for χ^2. Integrate $I_0^2 = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2\, e^{-\chi^2/2}$, where $\vec{\chi} = (x_1, x_2)$ in cylindrical coordinates to normalize the 2-d Gaussian, as in H01 2a). Show that this is the square of $I_0 = \int_{-\infty}^{\infty} dz\, e^{-z^2/2}$ for $\vec{\chi} = (z)$. Use this fact to normalize the 1-d Gaussian. [bonus: and by extension, any dimension]

b) Perform a change-of-variables to $\alpha x^2 = \chi^2/2$ on the above two integrals to evaluate $I_n(\alpha) = \int_0^{\infty} dx\, x^n e^{\alpha x^2}$ for $n = 0, 1$. Take the derivative with respect to α of both sides of the above integral indentities repeatedly to evaluate I_n for $n = 2, 3, \ldots$.

c) [bonus: Make the factorials hidden in b) explicit by transforming I_n to the Gamma function $\Gamma(\nu) = \int_0^{\infty} t^{\nu-1} e^{-t}$. As in a), show that $\Gamma(\nu+1) = \nu\Gamma(\nu)$, $\Gamma(0) = 1$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Thus $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$ and $\Gamma(\nu)$ is a generalization of the factorial to all real numbers.]

d) In 1-d, transform $z = (x - \mu)/\sigma$ and use the moments I_n to show that μ and σ are the *mean* and *standard deviation* of the Gaussian distribution, respectively.

2-d. Gaussian Generator (Reprise)—The general 2-d Gaussian, including both the *variances* σ_x^2, σ_y^2 and *covariance* $\sigma_{xy}^2 = \sigma_{yx}^2$, takes the form $p_G(x_1, x_2) = N e^{-\chi^2/2}$, where N is the normalization, $\vec{\chi} = \vec{x} - \vec{\mu} = (x_1 - \mu_1, x_2 - \mu_2)$, is the vector of *deviances*, and $\chi^2 = \vec{\chi} \cdot \vec{\chi} = \chi^T W \chi$ is *weighted* by the *metric* $W = \Sigma^{-1}$, which is the inverse of the symmetric *covariance matrix* $\Sigma = \begin{pmatrix} \sigma_x^2 & \sigma_{xy}^2 \\ \sigma_{yx}^2 & \sigma_y^2 \end{pmatrix}$.

If all covariances vanish (for example, in 1-d), this reduces to $\chi^2 = z_1^2 + z_2^2 + \ldots$ as previously discussed, where the variances are absorbed into $z_i = (x_i - \mu_i)/\sigma_i$, as before. In this case the distribution factorizes into the simpler product $p_G(z_1, z_2, \ldots) = p_G(z_1) p_G(z_2) \cdots$ of 1-d Gaussians. Otherwise, one must transform to new variables $\vec{\chi}' = (u_1, u_2, \ldots)$ to perform this factorization.

a) To generate a $p_G(x_1, x_2)$ with covariance, let $u = x + y$ and $v = x - y$ be independent. Generate n =100,000 random points (u, v) centered at $\vec{\mu} = (0, 0)$ with $\sigma_u = 1$ and $\sigma_v = 2$. Draw a *scatter plot* of (x, y). [bonus: draw the u and v axes on the same plot]

b) Calculate the means μ_x, μ_y, variances $\sigma_x^2 = \sigma_{xx}^2$, $\sigma_y^2 = \sigma_{yy}^2$, and covariance σ_{xy}^2, where $\sigma_{ij}^2 = \sum (x_i - \mu_i)(x_j - \mu_j)/n$, for $i, j = x, y$. Calculate the *correlation coefficient* $r = \sigma_{xy}/\sigma_x \sigma_y$.

c) Derive the distribution $p_G(x, y)$ from $p_G(u)$ and $p_G(v)$ used in a) to determine the covariance matrix Σ and compare with b). [bonus: graph the contours $\chi^2 = 1$ and $\chi^2 = 2$ in a)]

d) [bonus: plot a 2d histogram of (x, y) and calculate the χ^2 statistic on all bins with greater than 10 entries. What is the likelihood of the these random points following this distribution?]

e) [bonus: Describe the procedure for generating random pairs (x, y) from a general Gaussian distribution with means $\vec{\mu}$ and covariances Σ. How does this generalize to higher dimension?]