

University of Kentucky, Physics 335
Homework #3, Rev. C, due Thursday, 2021-11-03

1. The **asymmetry** $A = \frac{a-b}{a+b}$ is a convenient dimensionless parameter quantifying the *relative difference* between two positive observables a and b . It ranges from a maximum of $A = 1$ if $a \gg b$ to $A = 0$ if $a = b$ to the minimum value $A = -1$ if $a \ll b$. It is also used to describe the *polarization* $P = \frac{N_+ - N_-}{N_+ + N_-}$ of a two-state species with N_+ particles in one [spin] state and N_- in the other.

a) Calculate the uncertainty δA of A in terms of the covariance matrix $\delta^2 v_{ij}$ of $\vec{v} = (a, b)$ with elements $\delta_a^2 \equiv \text{Cov}(a, a) = \text{Var}(a)$, $\delta_{ab}^2 \equiv \text{Cov}(a, b) = \delta_{ba}^2$, and $\delta_b^2 \equiv \text{Cov}(b, b) = \text{Var}(b)$, using the error propagation formula $\delta^2 A_{\alpha\beta}(\vec{v}) = \sum_{i,j} \frac{\partial A_\alpha}{\partial v_i} \frac{\partial A_\beta}{\partial v_j} \delta^2 v_{ij}$ or $\delta^2 A = J \delta^2 v J^T$, where $J_{\alpha i} = \frac{\partial A_\alpha}{\partial v_i}$. For a single function $A(\vec{v})$, this reduces to the variance $\delta^2 A = \left(\frac{\partial A}{\partial a}\right)^2 \delta_a^2 + 2 \left(\frac{\partial A}{\partial a}\right) \left(\frac{\partial A}{\partial b}\right) \delta_{ab}^2 + \left(\frac{\partial A}{\partial b}\right)^2 \delta_b^2$.

b) For extra practice and to verify the consistency of error propagation, Build up the same formula piece by piece, deriving and using the following rules, including full covariance:

1. $\delta^2(u \pm v) = \delta_u^2 \pm 2\delta_{uv}^2 + \delta_v^2$, and $\delta^2(u + v, u - v) = \delta_u^2 - \delta_v^2$.
2. $\delta^2(\ln u) = \delta_u^2/u^2$ and $\delta^2(\ln u, \ln v) = \delta_{uv}^2/uv$.
3. $\delta^2(u \cdot v) / (u \cdot v)^2 = \delta_u^2/u^2 \pm 2\delta_{uv}^2/uv + \delta_v^2/v^2$ and $\delta^2(u \cdot v, u/v) = \delta_u^2/u^2 - \delta_v^2/v^2$ directly, and by combining 1 and 2. Note: $\cdot / \dots \pm$ is either $\cdot \dots +$ or $\cdot \dots -$. (*hint: keep all errors relative*)

c) Apply this formula to the polarization P , where the two particle counts N_\pm obey Poisson statistics: $\delta^2 N_\pm = N_\pm$, and are uncorrelated: $\delta^2 N_{+-} = 0$. Show that $N_\pm = \frac{1}{2}N(1 \pm P)$ and use these to express δP in terms of N and P .

2. If individual measurements $x_i \pm \delta x_i$ have different uncertainties δx_i , they should be combined using the **weighted average** $x = \sum_i w_i x_i / \sum_i w_i$ instead of the normal mean to emphasize the best data points, i.e. the ones with the smallest errors.

a) Calculate δx , assuming the weights have no error. Following Bevington, show that the weights w_i which minimize the combined error δx are proportional to $w_i = \delta^{-2} x_i$, and the proportionality constant cancels in the weighted average. How does the combined error relate to χ^2 ?

b) [bonus: Likewise, show that the *median* $x_{1/2}$ minimizes the unweighted *absolute deviation* $\alpha(m) = \sum_i |x_i - m|$. This is slightly different than Bevington's definition using the mean \bar{x} , but shows the connection between the median and absolute deviation. It also shows that the median is a *robust estimator* of central tendency because it minimizes a lower-order moment, and thus is less sensitive to outliers. The *mode* is the most robust of all, being completely unaffected by outliers as long as their frequency is less than that of the peak.]

c) Show for counts N_i obeying Poisson statistics: $\delta^2 N_i = N_i$, the sum $N = \sum_i N_i$ has variance $\delta^2 N = N$, which also obeys Poisson statistics.

d) Given n measurements of polarization $P_i = \frac{N_i^+ - N_i^-}{N_i^+ + N_i^-} \ll 1$, from individual counts N_i^\pm , each with $\delta^2 P_i$ from question 1c), show that optimal weight is $w_i = N_i$ and thus $P = \sum_i w_i P_i / \sum_i w_i = \frac{N^+ - N^-}{N^+ + N^-}$, where $N^\pm = \sum_i N_i^\pm$ is the sum of the total counts in each state. Thus it does not matter how you bin individual runs before calculating the total polarization.