University of Kentucky, Physics 335
Homework #5, Rev. B, due Wednesday, 2023-09-27

1-d: Average of Uniform Deviates—By the *Central Limit Theorem* (CLT), the average $\bar{x}$ of n samples from almost any random variable X tends to the Gaussian distribution as $n \to \infty$.

a) Calculate the mean μ and standard deviation σ of the *uniform* random variable X with probability distribution $p_u(x) = 1$ when $0 < x < 1$ and 0 otherwise.

b) Histogram the distribution of 100,000 averages of $n = 1$, 2, 3, 5, 10, and 100 uniform deviates each. Calculate the mean $\langle \bar{x} \rangle$ and standard deviation $s_{\bar{x}}$ of each sample distribution, and compare with $\mu_{\bar{x}} = \mu$, $\sigma_{\bar{x}} = \sigma/\sqrt{n}$ from the parent distribution. Plot the Gaussian distribution function $p_G(\bar{x}; \mu_{\bar{x}}, \sigma_{\bar{x}})$ together with each histogram.

c) Calculate $p_\chi(u; 1)$, the χ^2 distribution with $\nu = 1$ degree of freedom by using the *invariant* $p_\chi(u; 1)du = 2p_G(z)dz$, where $u = \chi^2 = z^2$. The factor of two comes from combining the left and right tails.

d) For the $\bar{x}$ data with $n = 100$ from part b), which approaches the Gaussian distribution, histogram $u = z^2$, where $z = (\bar{x} - \mu)/(\sigma_\mu) = (\bar{x} - \mu)/(\sigma/\sqrt{n})$, and plot together with the χ^2 distribution $p_\chi(u; 1)$ for $\nu = 1$ degree of freedom.

2-d: Gaussian Generator—The χ^2 distribution with $\nu = 2$ degrees of freedom can be used to create a more efficient and accurate Gaussian generator.

a) Integrate joint Gaussian distribution $p(\vec{\chi}) = p_G(z_1)p_G(z_2)$ of the vector $\vec{\chi} = (z_1, z_2) = (\chi \cos\phi, \chi \sin\phi)$ over ϕ in cylindrical coordinates (χ, ϕ) to obtain the χ^2 distribution $p_\chi(u; 2)$ where $u = \chi^2 = \vec{\chi} \cdot \vec{\chi} = {z_1}^2 + {z_2}^2$.

b) Histogram the distribution of $u = {z_1}^2 + {z_2}^2$ into 1000 bins over $0 \le u \le 10$, using 100,000 pairs (z_1, z_2) from 1b) with $n = 100$ again, and plot together with $p_\chi(u; 2)$ from 2a).

c) Working backwards, use the fact that $p_u(x)dx = p_e(u)du$ to determine the function $u(x)$ which transforms the uniform deviate x into u following the exponential distribution $p_e(u) = \frac{1}{2}\exp(-u/2)$ defined on $0 \le u < \infty$.

d) Generate a 100,000 pairs of Gaussian deviates $\vec{\chi} = (z_1, z_2)$ by generating $\chi = \sqrt{u}$ from the exponential deviate u of 2c) and a uniform deviate ϕ in the range $0 \le \phi < 2\pi$, and finally transforming (χ, ϕ) back to rectangular coordinates (z_1, z_2). Histogram both z's in 1000 bins over $-5 \le z < 5$ and plot each together with the normalized Gaussian distribution.

3-d: The **Maxwell distribution** of the energy of atoms in an ideal gas is just the χ^2 distribution for $\nu = 3$ degrees of freedom because each component of $\vec{v} = (v_x, v_y, v_z)$ is normally distributed according to its Boltzmann factor $p_B(v_x) = e^{-E/kT}$, where $E = v_x^2/2m$, etc.

a) Integrate the 3-d velocity distribution over spherical shells of thickness dv to derive the Maxwell distribution of velocities $p_M(v)$. Use the Gaussian generator of 2d) to generate 100,000 velocities in units of $\sqrt{mkT}$ and histogram them together with a plot of $p_M(v)$.

b) Equivalently, integrate the joint normal distribution $p(\vec{\chi}) = p_G(z_1)p_G(z_2)p_G(z_2)$ over spherical shells of radius du for $u = \vec{\chi} \cdot \vec{\chi}$ in the vector space $\vec{\chi} = (z_1, z_2, z_3)$ to obtain $p_\chi(u; 3)$. Thus, show that the Maxwell energy distribution equals $p_\chi(v^2/mkT; 3)$.

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***ν*-d:** [bonus: Higher dimensional χ^2 distributions are used to test the consistency of distributions, for example, in labs X03 and X04. Data are least-squares fit to a curve by minimizing a χ^2 function. In this case ν equals the number of Gaussian distributions (for example, the bins you are comparing) minus the number of constraints (for example, the parameters of a distribution or fit function).

a) Using the volume of a ν-dimensional ball $V_\nu(\chi) = \pi^{n/2}\chi^n/\Gamma(n/2+1)$, integrate the ν-d joint Gaussian distribution over hypershells to derive the χ^2 distribution $p_\chi(u; \nu)$. Plot the distribution for $\nu = 1, 2, 3, \ldots, 10$.

b) Show that as $\nu \to \infty$, $p_\chi(u; \nu)$ tends to the Gaussian distribution $p_G(u; \nu; 2\nu)$.]