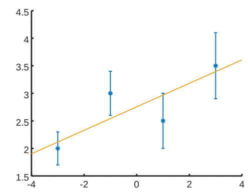
University of Kentucky, Physics 335
Homework #8, Rev. C, due Saturday, 2023-12-02

1. Linear fit. Fit the following data with uncertainties to a straight line:

i	x_i	$y_i \pm \delta y_i$
1	-3	20 ± 3
2	-1	30 ± 4
3	1	25 ± 5
4	3	35 ± 6



a) Calculate the $n \times m$ *design matrix* $X = [x^0\ x^1]$ for the linear fit $y = f(x) = a + bx$ for $n = 4$ data points to $m = 2$ fit parameters, and write out the corresponding matrix equation $Y = Xa$. Calculate the weight [metric] $W = \delta^{-2}Y$, (inverse covariance matrix), the least squares inverse $X^{\dashv} = (X^T W X)^{-1} X^T W$, the fit coefficients $a = X^{\dashv}Y$, and covariance matrix $\delta^2 a = (X^T W X)^{-1}$. Tabulate the residuals $\chi = Y - Xa$, individual squares $\chi_i^2 = ((y_i - f(x_i))/\delta y_i)^2$, and their sum $\chi^2 = \chi^T W \chi$. How many degrees of freedom ν are there?

b) [*bonus*: Neglecting the uncertainties for simplicity, solve for $X^{\dashv}$ by computing the Singular Value Decomposition (SVD) $X = UWV^T$ of the $n \times m$ matrix X, where U and V are orthogonal matrices of the *singular vectors* $U^T U = I_n$, $V^T V = I_m$, and W is the $m \times n$ diagonal matrix of *singular values* (eigenvalues) of X. Show algebraically that $X^{\dashv} = VW^{\dashv}U^T$, where $W^{\dashv}$ is the $m \times n$ diagonal matrix of inverse singular values. What is the geometric interpretation of the left and right singular vectors? Repeat with $\delta^2 Y$ using the Generalized SVD.]

2. The following **waveform** was captured on an oscilloscope with a Δt =1 ms sample interval, from the voltage output (mV) of a magnetometer measuring the field of an AC-driven coil:

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V= [3.81, 2.96, 2.36, 2.40, 0.47, 2.29, 0.51, -0.06, -2.00, -0.42, 0.06, ...
    0.57, 0.29, 2.07, 2.47, 2.86, 2.78, 1.86,  1.37, -0.43,  0.03]
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Fit these data for the amplitude A, angular frequency ω, phase ϕ, and offset B of a sine wave in the semilinear model $V(t) = A\cos(\omega t - \phi) + B$ ignoring errors using the following steps:

a) Plot the data and fit function, estimating of all four parameters (Note the data span approximately one period, ranging from the peak $V \approx 2$ mV, to the minimum $V \approx 0$ ms and back to the top). [*bonus*: estimate A, ϕ, ω, and B from the complex FFT of V.]

b) Linearize this model to $V = B + C\cos(\omega t) + D\sin(\omega t) = Ta$ for constant ω using the addition formula $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$. Follow the steps in 1a) to perform a linear fit on $V \approx Ta$ for parameters $a = (BCD)^T = T^{\dashv}V$, and calculate $\chi^2 = (V - Ta)^T(V - Ta)$.

c) Fit for ω by minimizing χ^2 as a function of ω. [*bonus*: fit $\chi^2(\omega)$ to a parabola] Calculate the fitted amplitude, phase, and offset from the corresponding fit parameters a and plot $V(t)$. Estimate the uncertainty δV of the voltage data by setting δV so that $\chi^2 = \nu = 21 - 4$ (number of degrees of freedom). [*bonus*: find covariance matrix of the fit parameters from $\delta^2 a$.]

c) [*bonus*: Fit the histogram of the residuals to a Gaussian $p_G(x) = N\exp(-((x-\mu)/\sigma)^2/2)$ by linearizing it to the quadratic form $\ln(p_G) = a + bx + cx^2$ and performing a linear fit on the binned χ_i^2 values given $(a\ b\ c)^T$. Compare your fitted values of μ and σ with the mean and residual of the distribution. Are they within error of what you expect, given χ^2 above? What is the likelihood of χ^2 being greater than the fitted value?]