University of Kentucky, Physics 335
Homework #5, Rev. A, due Monday, 2024-10-14

1-d. Gaussian Moments—In this homework we will learn the techniques for evaluating Gaussian intetegrals, and to use them to investigate the covariance of the joint Gaussian integral.

a) Let $N = \int_{-\infty}^{\infty} dz\, e^{-z^2/2}$ be the normalization of the standard 1-d Gaussian distribution. While this distribution cannot be integrated analytically, the 2-d Gaussian can, because its area element (Jacobian) in polar coordinates allows for a u-substitution. Show that it is equal to $N^2 = \int_{-\infty}^{\infty} dz_1 \int_{-\infty}^{\infty} dz_2\, e^{-\chi^2/2}$, where $\vec{\chi} = (z_1, z_2)$, and integrate in cylindrical coordinates (χ, ϕ), where $z_1 = \chi \cos\phi$ and $z_2 = \chi \sin\phi$, to get N and normalize the 1-d Gaussian. [*bonus*: what is the normalization of a ν-d Gaussian?]

b) Perform a change-of-variables on the above two integrals (over z and χ, respectively) to x, where $\alpha x^2 = z^2/2 = \chi^2/2$, to evaluate $I_n(\alpha) = \int_0^{\infty} dx\, x^n e^{-\alpha x^2}$ for $n = 0, 1$. Take the derivative with respect to α of both sides of the above integral indentities repeatedly to evaluate I_n for $n = 2, 3, 4$.

c) [*bonus*: Perform another change of variables to express each integral I_n in terms of the Gamma function $\Gamma(\nu) = \int_0^{\infty} t^{\nu-1} e^{-t} dt$. This shows the symmetry between even and odd n. Using the same technique as in b), show that $\Gamma(\nu+1) = \nu\Gamma(\nu)$, $\Gamma(1) = 1$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Thus $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$, and $\Gamma(\nu)$ is a generalization of the factorial to all real numbers.]

d) Transform the moments I_n from integrals over z to integrals over $x = \mu + \sigma z$ and show that μ and σ are the *mean* and *standard deviation*, respectively of the 1-d Gaussian distribution $p_G(x; \mu, \sigma)$.

1-d: Average of Uniform Deviates—By the *Central Limit Theorem* (CLT), the average $\bar{x}$ of n samples from almost any random variable X tends to the Gaussian distribution as $n \to \infty$.

a) Calculate the mean μ and standard deviation σ of the *uniform* random variable X with probability distribution $p_u(x) = 1$ when $0 < x < 1$ and 0 otherwise.

b) Histogram the distribution of 100,000 averages of $n = 1, 2, 3, 5, 10$, and 100 uniform deviates each. Calculate the mean $\langle \bar{x} \rangle$ and standard deviation $s_{\bar{x}}$ of each sample distribution, and compare with $\mu_{\bar{x}} = \mu$, $\sigma_{\bar{x}} = \sigma/\sqrt{n}$ from the parent distribution. Plot the Gaussian distribution function $p_G(\bar{x}; \mu_{\bar{x}}, \sigma_{\bar{x}})$ together with each histogram.

c) Calculate $p_\chi(u; 1)$, the χ^2 distribution with $\nu = 1$ degree of freedom by using the *invariant* $p_\chi(u; 1)du = 2p_G(z)dz$, where $u = \chi^2 = z^2$. The factor of two comes from combining the left and right tails.

d) For the $\bar{x}$ data with $n = 100$ from part b), which approaches the Gaussian distribution, histogram $u = z^2$, where $z = (\bar{x} - \mu)/(\sigma_\mu) = (\bar{x} - \mu)/(\sigma/\sqrt{n})$, and plot it together with the χ^2 distribution $p_\chi(u; 1)$ for $\nu = 1$ degree of freedom.