University of Kentucky, Physics 335
Homework #7, Rev. A, due Monday, 2024-10-28

1. The **asymmetry** $A = \frac{a-b}{a+b}$ is a convenient dimensionless parameter quantifying the *relative difference* between two positive observables a and b. It ranges from a maximum of $A = 1$ if $a \gg b$ to a minimum value $A = -1$ if $a \ll b$, and $A = 0$ if $a = b$. It is also used to describe the *polarization* $P = \frac{N_+ - N_-}{N_+ + N_-}$ of a two-state species with N_+ particles in one [spin] state and N_- in the other.

a) Calculate the uncertainty δA of A in terms of the covariance matrix $\delta^2 v_{ij}$ of $\vec{v} = (a, b)$ with elements $\delta_a^2 \equiv \text{Cov}(a, a) = \text{Var}(a)$, $\delta_{ab}^2 \equiv \text{Cov}(a, b) = \delta_{ba}^2$, and $\delta_b^2 \equiv \text{Cov}(b, b) = \text{Var}(b)$, using the error propagation formula $\delta^2 A_{\alpha\beta}(\vec{v}) = \sum_{i,j} \frac{\partial A_\alpha}{\partial v_i} \frac{\partial A_\beta}{\partial v_j} \delta^2 v_{ij}$ or $\delta^2 A = J \delta^2 v J^T$, where $J_{\alpha i} = \frac{\partial A_\alpha}{\partial v_i}$. For a single function $A(\vec{v})$, this reduces to the variance $\delta^2 A = \left(\frac{\partial A}{\partial a}\right)^2 \delta_a^2 + 2\left(\frac{\partial A}{\partial a}\right)\left(\frac{\partial A}{\partial b}\right)\delta_{ab}^2 + \left(\frac{\partial A}{\partial b}\right)^2 \delta_b^2$.

b) For extra practice and to verify the consistency of error propagation, we will now calculate the same formula in part c), deriving and using the following rules, which include covariance:

1. $\delta^2(u \pm v) = \delta_u^2 \pm 2\delta_{uv}^2 + \delta_v^2$, and $\delta^2(u + v, u - v) = \delta_u^2 - \delta_v^2$.
2. $\delta^2(\ln u) = \delta_u^2/u^2$ and $\delta^2(\ln u, \ln v) = \delta_{uv}^2/uv$.
3. $\delta^2(uv)/(uv)^2 = \delta_u^2/u^2 + 2\delta_{uv}^2/uv + \delta_v^2/v^2$, $\delta^2(u/v)/(u/v)^2 = \delta_u^2/u^2 - 2\delta_{uv}^2/uv + \delta_v^2/v^2$, and $\delta^2(uv, u/v)/u^2 = \delta_u^2/u^2 - \delta_v^2/v^2$ directly from $\delta^2(uv)$, etc., and by applying rules 1 and 2 to the error of both sides of $\ln(uv) = \ln(u) + \ln(v)$ and $\ln(u/v) = \ln(u) - \ln(v)$.

c) Apply this formula to the polarization P, where the particle counts N_+ and N_- obey Poisson statistics: $\delta^2 N_\pm = N_\pm$, and are uncorrelated: $\delta^2 N_{+-} = 0$. Show that $N_\pm = \frac{1}{2}N(1 \pm P)$, where $N = N_+ + N_-$, and thus express δP in terms of N and P.

2. If individual measurements $x_i \pm \delta x_i$ have different uncertainties δx_i, they should be combined using the **weighted average** $x = \sum_i w_i x_i / \sum_i w_i$ instead of the normal mean to emphasize the best data points, i.e. the ones with the smallest errors.

a) Calculate δx, assuming the weights have no error. Following Bevington, show that the weights w_i which minimize the combined error δx are propotional to $w_i = \delta^{-2} x_i$, and the proportionality constant cancels in the weighted average. [*bonus:* How does the combined error relate to χ^2?]

b) [*bonus:* Likewise, show that the *median* $x_{1/2}$ minimizes the unweighted *absolute deviation* $\alpha(m) = \sum_i |x_i - m|$. This is slightly different than Bevington's definition using the mean $\bar{x}$, but shows the connection between the median and absolute deviation. It also shows that the median is a *robust estimator* of central tendency because it minimizes a lower-order moment, and thus is less sensitive to outliers. The *mode* is the most robust of all, being completely unaffected by outliers as long as their frequency is less than that of the peak.]

c) Show for counts N_i obeying Poisson statistics: $\delta^2 N_i = N_i$, the sum $N = \sum_i N_i$ has variance $\delta^2 N = N$, also obeying Poisson statistics, which is an important self-consistency check.

d) Given n measurements of polarization $P_i = \frac{N_i^+ - N_i^-}{N_i^+ + N_i^-} \ll 1$, from individual counts $N_i^\pm$, each with $\delta^2 P_i$ from question 1c), show that optimal weight is $w_i = N_i$ and thus $P = \sum_i w_i P_i / \sum_i w_i = \frac{N^+ - N^-}{N^+ + N^-}$, where $N^\pm = \sum_i N_i^\pm$ is the sum of the total counts in each state. Thus it does not matter how you bin individual measurements before calculating the total polarization or asymmetry.