HW1 Solutions

1a) $$\int_{-\infty}^{\infty} f(x)\,dx = C\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}dx$$

let $u = \frac{x-\mu}{\sigma}$

$du = \frac{dx}{\sigma}$

$\alpha = \frac{1}{2}$

$$= C\cdot\sigma\int_{-\infty}^{\infty} e^{-\alpha u^2}du$$

$$= C\cdot\sigma\cdot\sqrt{\frac{\pi}{\alpha}} = 1$$

so $C = \frac{1}{\sigma\sqrt{2\pi}}$

b) $$\langle x\rangle \equiv \frac{\int_{-\infty}^{\infty} x\cdot f(x)\,dx}{\underbrace{\int_{-\infty}^{\infty} f(x)\,dx}_{=1}} = C\int_{-\infty}^{\infty} \underbrace{(x-\mu)e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}_{\text{odd function integral }=0.}dx + \mu\underbrace{\int_{-\infty}^{\infty} f(x)\,dx}_{=1} = \mu$$

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c) $$\frac{dI}{d\alpha} = \int_{-\infty}^{\infty}\frac{d}{d\alpha}e^{-\alpha u^2}du = \int_{-\infty}^{\infty} -u^2 e^{-\alpha u^2}du = \frac{d}{d\alpha}\sqrt{\frac{\pi}{\alpha}} = -\frac{1}{2}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\langle x^2\rangle \equiv \frac{\int_{-\infty}^{\infty} x^2 f(x)\,dx}{\underbrace{\int_{-\infty}^{\infty} f(x)\,dx}_{=1}} = \int_{-\infty}^{\infty}\left[\sigma^2\left(\frac{x-\mu}{\sigma}\right)^2 + \underset{\text{part (b)}}{2\mu x} - \underset{\text{part(a)}}{\mu^2}\right]f(x)\,dx$$

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$$= \sigma^3\int_{-\infty}^{\infty} u^2\cdot Ce^{-\frac{1}{2}u^2}du + 2\mu\langle x\rangle - \mu^2$$

$$= \sigma^3\cdot C\cdot\frac{1}{2}\sqrt{\frac{\pi}{(1/2)^3}} + \mu^2 = \sigma^2 + \mu^2 = \sigma^2 + \langle x\rangle^2$$

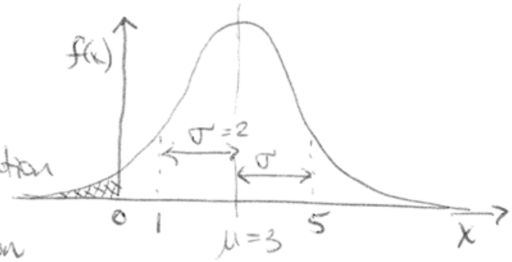
$$\langle(x-\langle x\rangle)^2\rangle = \int_{-\infty}^{\infty}(x-\langle x\rangle)^2 f(x)\,dx = \int_{-\infty}^{\infty}\left(x^2 - 2x\langle x\rangle + \langle x\rangle^2\right)f(x)\,dx$$

5 (bonus)

$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx - 2\langle x\rangle\int_{-\infty}^{\infty} x\,f(x)\,dx + \langle x\rangle^2\int_{-\infty}^{\infty} f(x)\,dx$$

$$= \langle x^2\rangle - 2\langle x\rangle\cdot\langle x\rangle + \langle x\rangle^2 = \langle x^2\rangle - \langle x\rangle^2$$

1.d) μ = mean, x-value at the center of the distribution



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σ = RMS width, standard deviation

$$P(x<0) = \int_{-\infty}^{0} f(x)\,dx = \frac{1}{2} - \int_{0}^{3}\frac{1}{\sqrt{8\pi}}e^{-\frac{1}{2}\left(\frac{x-3}{2}\right)^2}dx$$

$$= \frac{1}{2} - 0.433 = \boxed{6.7\%}$$

2008-02-02

HW1 Solutions

1.

$$v_{RMS} = \sqrt{\frac{3kT}{m}} = \left(\frac{3(1.38\times10^{-23}\,J/K)(300K)}{2\cdot1.008\cdot1.66\times10^{-27}kg}\right)^{1/2} = 1.93\,km/s$$

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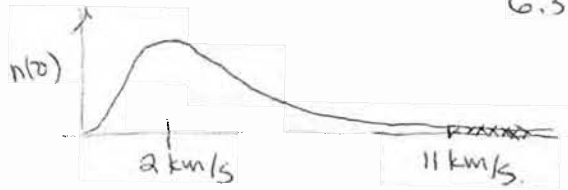
$$v_{escape} = \sqrt{\frac{2GM}{R}} = \left(\frac{2(6.67\times10^{-11}\,kg^{-1}m^3/s^2)(6.0\times10^{24}kg)}{6.371\times10^6 m}\right)^{1/2} = 11.2\,km/s$$



The atoms in the tail have enough kinetic energy to escape. After the H_2 rethermalizes, the velocities will be redistributed with other atoms in the tail, which also escape, etc...

(this also explains evaporative cooling)

2.

T=300K

	H_2	He	O_2	N_2	
A	2×1.008	4.003	2×16.00	2×14.01	u
v_{rms}	1.93	1.37	0.484	0.517	km/s
$T=\frac{1}{2}mv_{rms}^2$	$=\frac{3}{2}kT = 6.21\times10^{-21}J = 38.8$ meV				

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By the equipartition theorem, all energies are the same.

The velocity (RMS) decreases as the mass increases.

3.

$$\tilde{u}(\nu)d\nu = \frac{8\pi\nu^2}{c^3}\frac{h\nu\, d\nu}{e^{h\nu/kT}-1} = u(\lambda)d\lambda$$

$$= \frac{8\pi\left(\frac{c}{\lambda}\right)^2}{c^3}\frac{\frac{hc}{\lambda}\left(\frac{-c}{\lambda^2}d\lambda\right)}{e^{hc/\lambda kT}-1}$$

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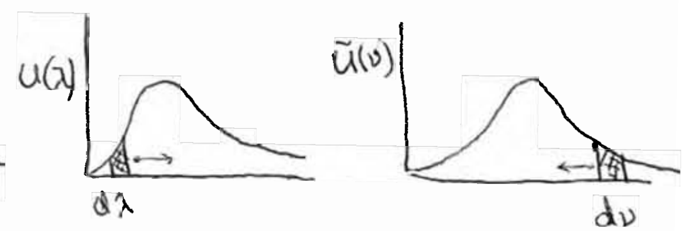
$$= \cancel{-}\,\frac{8\pi hc}{\lambda^5}\frac{1}{e^{hc/\lambda kT}-1}$$

$$\nu = \frac{c}{\lambda}$$
$$d\nu = \frac{-c}{\lambda^2}d\lambda$$



The areas should be the same.

The (-) sign was because $\nu\to0$ as $\lambda\to\infty$. Ignore it.

4.

$$R = \frac{1}{4}c\int_0^\infty u\,d\lambda = \frac{1}{4}c\int_0^\infty \frac{8\pi hc\,\lambda^{-5}d\lambda}{e^{hc/\lambda kT}-1}$$

let $x = \frac{hc}{\lambda kT}$

$$dx = -\frac{hc\,d\lambda}{\lambda^2 kT}$$

$$= \frac{1}{4}c\cdot8\pi hc\int_0^\infty \frac{\left(\frac{kTx}{hc}\right)^{5-2}\frac{kT\,dx}{hc}}{e^x-1}$$

$$\to \int_0^\infty d\lambda = \int_\infty^0 -dx = \int_0^\infty dx$$

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$$= \frac{1}{4}c\cdot\frac{8\pi(kT)^4}{(hc)^5}\underbrace{\int_0^\infty\frac{x^3dx}{e^x-1}}_{\pi^4/15} = \underbrace{\frac{2\pi^5k^4}{15h^3c^2}}_{\sigma}T^4$$

$$\sigma = \frac{2\cdot\pi^5\cdot(1.38\times10^{-23}\,J/K)^4}{15(6.63\times10^{-34}\,J\cdot s)^3(3.00\times10^8\,m/s)^2} = 5.67\times10^{-8}\frac{W}{m^2K^4}$$

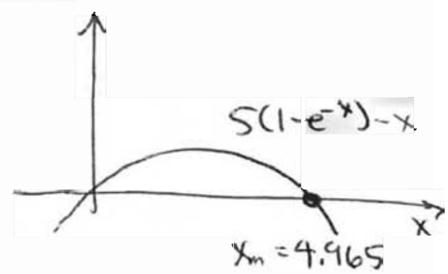
note: $R = 459\,W/m^2$ at 300K!

5. $u = \dfrac{8\pi hc\lambda^{-5}}{e^{hc/\lambda kT}-1} = \underbrace{8\pi hc\left(\dfrac{kT}{hc}\right)^5}_{\text{const.}} \dfrac{x^5}{e^x-1} = \tilde{u}(x)$

Maximizing $u(\lambda)$ is equivalent to maximizing $\tilde{u}(x)$ w/r x by the chain rule. $u'(\lambda) = \tilde{u}'(x)\dfrac{dx}{d\lambda} = 0$

Ignore the constant. $\rightarrow u'(x) = 0$

$$\frac{d\tilde{u}}{dx} = \frac{d}{dx}\frac{x^5}{e^x-1} = \frac{(e^x-1)(5x^4)-(x^5)(e^x)}{(e^x-1)^2} = 0$$



$$\left(5(e^x-1) - e^x x\right)x^4 = 0$$

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$$5(1-e^{-x}) - x = 0$$

note as x gets large, $e^{-x} \to 0$

so $x \approx 5$. By recursion,

$$x_1 \approx (5)(1-e^{-5}) = 4.966$$
$$x_2 = (x_1)(1-e^{-x_1}) \approx 4.965$$

so $\boxed{\lambda_m T = \dfrac{hc}{xk} = 2.898 \text{ mK}\cdot\text{m}}$

thus sun: $T = 5800K$ $\lambda_m = \dfrac{2.898 \text{ mK}\cdot\text{m}}{5800K} = 515 \text{ nm} = $ green

the color of the sun is an average of blue-green-yellow.