Problem 1

The normal distribution (Gaussian distribution or bell curve) has the form

$$f(x) = Ce^{-\frac{1}{2}\left(\frac{(x-\mu)}{\sigma}\right)^2}$$

Part a

Calculate the normalization factor C by requiring the distribution to be normalized

$$\int_{x=-\infty}^{\infty} f(x)\,dx = 1$$

Substituting in $f(x)$, defined above, this becomes:

$$\int_{-\infty}^{\infty} Ce^{-\frac{1}{2}\left(\frac{(x-\mu)}{\sigma}\right)^2}\,dx = 1$$

If we set $b = \frac{x-\mu}{\sigma}$, and thus $db = \frac{dx}{\sigma}$, then we can substitute $x = b\sigma + \mu$ and $dx = \sigma\,db$ into the equation. It becomes:

$$\int_{-\infty}^{\infty} Ce^{-\frac{1}{2}b^2}\sigma\,db = 1$$

Using $I_n = \int_0^\infty x^n e^{-\lambda x^2}\,dx$ with $n = 0$. According to [Tipler & Llewellyn, p AP 16-17], in this case $I_0 = \frac{1}{2}\pi^{1/2}\lambda^{-1/2}$. Since $e^{-\lambda x^2}$ is an even function, integrating from $-\infty$ to ∞ gives $2I_0 = \pi^{1/2}\lambda^{-1/2}$. In our case, $\lambda = -\frac{1}{2}$. So we have:

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} Ce^{-\frac{1}{2}b^2}\sigma\,db \\
&= \sigma C\int_{-\infty}^{\infty} e^{-\frac{1}{2}b^2}\,db \\
&= \sigma C\pi^{1/2}\left(\frac{1}{2}\right)^{-1/2} \\
1 &= C\sigma\sqrt{2\pi}
\end{aligned}$$

So, $C = \frac{1}{\sigma\sqrt{2\pi}}$.

Part b

Calculate $\langle x\rangle$, the expected value of x, defined by

$$\langle u\rangle = \frac{\int_{-\infty}^{\infty} uf(x)\,dx}{\int_{-\infty}^{\infty} f(x)\,dx}$$

What physical interpretation does it have?

Since $\int_{-\infty}^{\infty} f(x)\,dx = 1$, as defined in Part a, and substituting in x for u, we get:

$$\langle x\rangle = \int_{-\infty}^{\infty} xf(x)\,dx$$

Substituting in for $f(x)$, we get:

$$\langle x\rangle = \int_{-\infty}^{\infty} xCe^{-\frac{1}{2}\left(\frac{(x-\mu)}{\sigma}\right)^2}\,dx$$

Once again, if we set $b = \frac{x-\mu}{\sigma}$, and thus $db = \frac{dx}{\sigma}$, then we can substitute $x = b\sigma + \mu$ and $dx = \sigma\,db$ into the equation. It becomes:

$$\begin{aligned}
\langle x\rangle &= \int_{-\infty}^{\infty} C(b\sigma+\mu)e^{-\frac{1}{2}b^2}\sigma\,db \\
&= \int_{-\infty}^{\infty} Cb\sigma^2e^{-\frac{1}{2}b^2}\,db + \int_{-\infty}^{\infty} C\sigma\mu e^{-\frac{1}{2}b^2}\,db \\
&= C\sigma^2\int_{-\infty}^{\infty} be^{-\frac{1}{2}b^2}\,db + \mu\int_{-\infty}^{\infty} Ce^{-\frac{1}{2}b^2}\sigma\,db
\end{aligned}$$

From Part a, we know that $\int_{-\infty}^{\infty} Ce^{-\frac{1}{2}b^2}\sigma\,db = 1$. [Tipler & Llewellyn, p AP 16-17] also tell us many wonderful things about how to integrate $\int_0^\infty xe^{-\lambda x^2}\,dx$, however, we are integrating from $-\infty$ to ∞, and $xe^{-\lambda x^2}$ is an odd function, so the first integral is zero. So we get:

$$\langle x\rangle = \mu$$

This represents the mean of the distribution.

Part c

Calculate $\langle x^2 \rangle$. Why is this different than $\langle x \rangle^2$? Express σ in terms of $\langle x \rangle$ and $\langle x^2 \rangle$. What physical interpretation does it have? Show that

$$\langle (x - \langle x \rangle)^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2$$

We know $\langle x \rangle = \mu$, so we need to calculate $\langle x^2 \rangle$. Substituting in $u = x^2$ into the equation in Part b, we get

$$\langle x^2 \rangle = \frac{\int_{-\infty}^{\infty} x^2 f(x)\, dx}{\int_{-\infty}^{\infty} f(x)\, dx}$$

Once again, $\int_{-\infty}^{\infty} f(x)\, dx = 1$, so we have:

$$\begin{aligned}
\langle x^2 \rangle &= \int_{-\infty}^{\infty} x^2 f(x)\, dx \\
&= \int_{-\infty}^{\infty} x^2 C e^{-\frac{1}{2}\left(\frac{(x-\mu)}{\sigma}\right)^2} dx \\
&= C \int_{-\infty}^{\infty} x^2 e^{-\frac{1}{2}\left(\frac{(x-\mu)}{\sigma}\right)^2} dx
\end{aligned}$$

And yet again, if we set $b = \frac{x-\mu}{\sigma}$, and thus $db = \frac{dx}{\sigma}$, then we can substitute $x = b\sigma + \mu$ and $dx = \sigma\, db$ into the equation. It becomes:

$$\begin{aligned}
\langle x^2 \rangle &= C \int_{-\infty}^{\infty} (b\sigma + \mu)^2 e^{-\frac{1}{2}b^2} \sigma\, db \\
&= C \int_{-\infty}^{\infty} (b^2\sigma^3 + 2b\sigma^2\mu + \mu^2\sigma) e^{-\frac{1}{2}b^2}\, db \\
&= C\sigma^3 \int_{-\infty}^{\infty} b^2 e^{-\frac{1}{2}b^2}\, db \\
&\quad + 2C\sigma^2\mu \int_{-\infty}^{\infty} b e^{-\frac{1}{2}b^2}\, db \\
&\quad + \mu^2 \int_{-\infty}^{\infty} C e^{-\frac{1}{2}b^2} \sigma\, db
\end{aligned}$$

Similar to Part b, the integral in the third term is equal to one, leaving the third term as μ^2. The integral in the second term is for an odd function from $-\infty$ to ∞, so it is equal to zero. The integral in the first term, however, resembles I_2 from [Tipler & Llewellyn, p AP 16-17], except that we are integrating from $-\infty$ to ∞, instead of from 0 to ∞. This gives us$2I_2 = \frac{1}{2}\pi^{1/2}\lambda^{-3/2}$, which gives:

$$C\sigma^3 \int_{-\infty}^{\infty} b^2 e^{-\frac{1}{2}b^2}\, db = \frac{C\sigma^3\sqrt{\pi}}{2\left(\frac{1}{2}\right)^{3/2}}$$

But, $C = \frac{1}{\sigma\sqrt{2\pi}}$, and substituting this in we get:

$$\frac{C\sigma^3\sqrt{\pi}}{\left(\frac{1}{2}\right)^{1/2}} = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)\sigma^3\sqrt{2\pi} = \sigma^2$$

So, $\langle x^2 \rangle = \sigma^2 + \mu^2$.

To find σ in terms of $\langle x \rangle$ and $\langle x^2 \rangle$, we can substitute $\mu = \langle x \rangle$ into $\langle x^2 \rangle = \sigma^2 + \mu^2$, and we get:

$$\begin{aligned}
\langle x^2 \rangle &= \sigma^2 + \langle x \rangle^2 \\
\sigma^2 &= \langle x^2 \rangle - \langle x \rangle^2 \\
\sigma &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2}
\end{aligned}$$

This gives us the population standard deviation of the distribution.

$\langle x^2 \rangle$ is also x^2_{RMS}, which is $\frac{x_1^2 + x_2^2 + \cdots + x_N^2}{N}$. $\langle x \rangle^2$, on the other hand, is $\left(\frac{x_1 + x_2 + \cdots + x_N}{N}\right)^2$, which not only has multiple cross terms of x_i in the numerator, but also has N^2 in the denominator. So, $\langle x \rangle^2$ is clearly not equal to $\langle x^2 \rangle$.

To determine if

$$\langle (x - \langle x \rangle)^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2$$

we check to see what each side equals. From previous calculations, we have:

$$\langle x^2\rangle - \langle x\rangle^2 = \sigma^2 + \mu^2 - \mu^2 = \sigma^2$$

and, by the definition of $\langle u\rangle$ given above:

$$\begin{aligned}
\left\langle (x - \langle x\rangle)^2 \right\rangle &= \int_{-\infty}^{\infty} (x - \langle x\rangle)^2 f(x)\,dx \\
&= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx \\
&= \int_{-\infty}^{\infty} \left(x^2 - 2x\mu + \mu^2\right) f(x)\,dx \\
&= \int_{-\infty}^{\infty} x^2 f(x)\,dx \\
&\quad - \int_{-\infty}^{\infty} 2x\mu f(x)\,dx \\
&\quad + \int_{-\infty}^{\infty} \mu^2 f(x)\,dx \\
&= \int_{-\infty}^{\infty} x^2 C e^{-\frac{1}{2}\left(\frac{(x-\mu)}{\sigma}\right)^2} dx \\
&\quad -2\mu \int_{-\infty}^{\infty} x C e^{-\frac{1}{2}\left(\frac{(x-\mu)}{\sigma}\right)^2} dx \\
&\quad +\mu^2 \int_{-\infty}^{\infty} C e^{-\frac{1}{2}\left(\frac{(x-\mu)}{\sigma}\right)^2} dx
\end{aligned}$$

However, we have already calculated all three of these integrals. From earlier in this problem, we found that $\int_{-\infty}^{\infty} x^2 C e^{-\frac{1}{2}\left(\frac{(x-\mu)}{\sigma}\right)^2} dx = \sigma^2 + \mu^2$, from part (b) we know that $\int_{-\infty}^{\infty} x C e^{-\frac{1}{2}\left(\frac{(x-\mu)}{\sigma}\right)^2} dx = \mu$, and, by definition, we know that $\int_{-\infty}^{\infty} C e^{-\frac{1}{2}\left(\frac{(x-\mu)}{\sigma}\right)^2} dx = 1$. So, we're left with:

$$\begin{aligned}
\left\langle (x - \langle x\rangle)^2 \right\rangle &= \sigma^2 + \mu^2 - 2\mu(\mu) + \mu^2(1) \\
&= \sigma^2
\end{aligned}$$

Since both sides of the equation are equal to σ^2, $\left\langle (x - \langle x\rangle)^2 \right\rangle = \langle x^2\rangle - \langle x\rangle^2$. Note that it is also possible to use the properties associated with averaging to solve this problem:

$$\begin{aligned}
\left\langle (x - \langle x\rangle)^2 \right\rangle &= \left\langle x^2 - 2x\langle x\rangle + \langle x\rangle^2 \right\rangle \\
&= \langle x^2\rangle - \langle 2x\langle x\rangle\rangle + \left\langle \langle x\rangle^2 \right\rangle
\end{aligned}$$

From earlier, we know that $\langle x^2\rangle = \sigma^2 + \mu^2$ and that $\langle x\rangle = \mu$, so we have:

$$\left\langle (x - \langle x\rangle)^2 \right\rangle = \sigma^2 + \mu^2 - \langle 2x\mu\rangle + \langle \mu^2\rangle$$

But, 2μ and μ^2 are constants, and can be pulled out of the averages:

$$\begin{aligned}
\left\langle (x - \langle x\rangle)^2 \right\rangle &= \sigma^2 + \mu^2 - 2\mu\langle x\rangle + \mu^2\langle 1\rangle \\
&= \sigma^2 + \mu^2 - 2\mu^2 + \mu^2 \\
&= \sigma^2
\end{aligned}$$

Part d

Given $\mu = 3$ and $\sigma = 2$, calculate the probability that $x < 0$. Note: there is no analytic formula for the result, so you will have to calculate the integral numerically or look it up in a table.

We can calculate $z = \frac{|x-\mu|}{\sigma} = \frac{|0-3|}{2} = 1.5$ when $x = 0$. The probability of getting a value of x between $\mu - z\sigma = 3 - 3 = 0$ and $\mu + z\sigma = 3 + 3 = 6$ is 0.86638, according to Table C.2 in [Bevington & Robinson, p251]. Thus, the chance of getting a value of x outside this range is $1 - 0.86638 = 0.13362$, and the probability of getting values only below zero should be half this probability (since it represents half of the remaining distribution), or $0.13362/2 = 0.06681$.

Problem 2

Calculate $v_{RMS} \equiv \sqrt{\langle v^2\rangle}$ for H_2 molecules at $T = 300\,\mathrm{K}$. Using potential energy, show that the escape velocity of the earth's graitational field is $v_{escape} = \sqrt{2GM/R}$. Compare this vallue with v_{RMS}. The earth's atmosphere contains very little H_2. How is it possible that the H_2 the molecules escape with a relatively small v_{RMS}? Why doesn't N_2 escape? [Tipler & Llewellyn, p8]

We know that $v_{RMS} = \sqrt{\frac{3kT}{m}}$, where T is the temperature, m is the mass of the particle, and $k = 1.381 \times 10^{-23}$J/K. Substituting, we have:

$$\begin{aligned} v_{RMS} &= \sqrt{\frac{3\,(1.381 \times 10^{-23}\text{J/K})\,(300\ \text{K})}{3.34 \times 10^{-27}\text{kg}}} \\ &\approx 1929\ \text{m/s} \end{aligned}$$

The escape velocity is the velocity at which some object has sufficient kenetic energy to overcome the gravational attraction of some other object. That is, at infinite distance, the speed of an object initial traveling at escape velocity will be essentially zero, but the escaping object will be able to reach this distance, having overcome the gravational pull of the reference object. So, $T - U = 0$. If $T = \frac{1}{2}mv_e^2$, where v_e is the escape velocity and m is the mass of the escaping object, and $U = \frac{GMm}{r}$, where $G = 6.673 \times 10^{-11}\frac{\text{m}^3}{\text{kg}\cdot\text{s}^2}$ is the gravitional constant, M is the mass of the reference object, m is the mass of the escaping object, and r is the initial distance between the center of the reference object and the escaping object, then we can easily calculate v_e:

$$\begin{aligned} T - U &= 0 \\ \frac{1}{2}mv_e^2 - \frac{GMm}{r} &= 0 \\ \frac{1}{2}mv_e^2 &= \frac{GMm}{r} \\ v_e^2 &= \frac{2GM}{r} \\ v_e &= \sqrt{\frac{2GM}{r}} \end{aligned}$$

If one of these objects is the earth, with mass $M = 5.9742 \times 10^{24}$kg [Google] and radius $r = 6378.1$ km [also Google], then the escape velocity for the earth is:

$$\begin{aligned} v_e &= \sqrt{\frac{2GM}{r}} \\ &= \sqrt{\frac{2\left(6.673 \times 10^{-11}\frac{\text{m}^3}{\text{kg}\cdot\text{s}^2}\right)(5.9742 \times 10^{24}\text{kg})}{6378.1 \times 10^3\,\text{m}}} \\ &\approx 11,181\,\text{m/s} \end{aligned}$$

This escape velocity is much larger than the v_{RMS} of H_2. However, it is still possible for the hydrogen molecules to escape because the speed of hydrogen molecules in the atmosphere follows Maxwellian distribution and is not, in fact, a homogenous medium of hydrogen molecules all traveling at a speed of v_{RMS}. Some molecules in this distribution will have enough kinetic energy to overcome Earth's gravity. Nitrogen will also follow a Maxwellian distribution, but the v_{RMS} of N_2 molecules at 300 K is:

$$\begin{aligned} v_{RMS} &= \sqrt{\frac{3\,(1.381 \times 10^{-23}\text{J/K})\,(300\ \text{K})}{4.68 \times 10^{-26}\text{kg}}} \\ &\approx 515\ \text{m/s} \end{aligned}$$

This is significantly smaller than the v_{RMS} of hydrogen molecules. While it is still possible for individual N_2 molecules to escape Earth's atmosphere, the number of nitrogen molecules that are capable of attaining the requisite escape velocity must be significantly lower than for hydrogen molecules. According to [Tipler & Llewellyn, p323], a gas will escape from a planet's atmosphere in 10^8 years if the average speed of its molecules is one-sixth of the escape velocity. The average velocity of nitrogen molecules is, also according to [Tipler & Llewellyn, p321], about 475 m/s. This is roughly 1/24th the escape velocity for Earth. I do not know if the 10^8 figure would scale up naïvely, so that the time required would be, say, $10^{8*4} = 10^{24}$ years, but if this is the case, then the earth would have to be 10^{13} billion years old in order for N_2 gas to completely escape and, according to Wikipedia, the age of the earth is only about 4.5 billion years old. Even if the 10^8 figure scaled up as $10^{8+4} = 10^{12}$, this is still 100 billion years. Earth still is not nearly old enough for all the nitrogen molecules to have all escaped. Also, it may be that gasses of molecules with an average velocity of less than 1/6th Earth's escape velocity will never able to entirely escape Earth's gravational field. The book was not entirely clear on this.

Problem 3

> Compare v_{RMS} and the average kinetic energy per atom/molecule for H_2, He, O_2, and N_2 gas under standard temperature and pressure.

We already know from Problem 2 that the v_{RMS} of H_2 is about 1929 m/s, and for N_2 is about 515 m/s. For O_2 and He, we first have to calculate the mass of the individual molecules in units that are actually usable. This is done by dividing the molar/atomic mass by $N_A = 6.022 \times 10^{23}$, Avagadro's number:

$$m_{\text{He}} \approx \left(\frac{4.0026\,\text{g/mole}}{6.022 \times 10^{23}\text{particles/mole}}\right)\left(\frac{1\,\text{kg}}{1000\,\text{g}}\right) \approx 6.647 \times 10^{-27}\text{kg}$$

$$m_{\text{O}_2} \approx \left(\frac{32\,\text{g/mole}}{6.022 \times 10^{23}\text{particles/mole}}\right)\left(\frac{1\,\text{kg}}{1000\,\text{g}}\right) \approx 5.314 \times 10^{-26}\text{kg}$$

Now we can calculate v_{RMS} for helium atoms and oxygen molecules exactly as was done for nitrogen and hydrogen molecules:

$$v_{RMS}\,\text{He} \approx \sqrt{\frac{3\,(1.381 \times 10^{-23}\text{J/K})\,(300\ \text{K})}{6.647 \times 10^{-27}\text{kg}}} \approx 1367\ \text{m/s}$$

$$v_{RMS}\,\text{O}_2 \approx \sqrt{\frac{3\,(1.381 \times 10^{-23}\text{J/K})\,(300\ \text{K})}{5.314 \times 10^{-26}\text{kg}}} \approx 484\ \text{m/s}$$

The average kinetic energy, then, of each of these gasses is just $\langle E\rangle = \left\langle \frac{1}{2}mv^2\right\rangle = \frac{1}{2}m\left\langle v^2\right\rangle = \frac{1}{2}mv_{RMS}^2$. So we have:

$$\begin{aligned}
\left\langle E_{\text{H}_2}\right\rangle &\approx \frac{1}{2}\left(3.34 \times 10^{-27}\text{kg}\right)(1929\ \text{m/s})^2 \\
&\approx 6.21 \times 10^{-21}\,\text{J} \\
\left\langle E_{\text{N}_2}\right\rangle &\approx \frac{1}{2}\left(4.68 \times 10^{-26}\text{kg}\right)(515\ \text{m/s})^2 \\
&\approx 6.21 \times 10^{-21}\,\text{J} \\
\left\langle E_{\text{He}}\right\rangle &\approx \frac{1}{2}\left(6.647 \times 10^{-27}\text{kg}\right)(1367\ \text{m/s})^2 \\
&\approx 6.21 \times 10^{-21}\,\text{J} \\
\left\langle E_{\text{O}_2}\right\rangle &\approx \frac{1}{2}\left(5.314 \times 10^{-27}\text{kg}\right)(484\,m/s)^2 \\
&\approx 6.22 \times 10^{-21}\,\text{J}
\end{aligned}$$

So, while the various v_{RMS} values are significantly different, the kinetic energies for each molecule at standard temperature and pressure are basically identical. This stands to reason, since, according to [Tipler & Llewellyn, p324], the average energy of a molecule is independent of mass. This could have been calculated for each atom/molecule by using the formula:

$$\begin{aligned}
\langle E\rangle &= \frac{3}{2}kT \\
&\approx \frac{3}{2}(1.38 \times 10^{-23})(300) \\
&\approx 6.21 \times 10^{-21}\,\text{J}
\end{aligned}$$

Problem 4

> Show that the following blackbody distributions written in terms of wavelength or frequency are equivalent:

$$\tilde{u}(f)\,df = \frac{8\pi f^2}{c^3}\frac{hf}{e^{\frac{hf}{kT}}-1}\,|df|$$

$$u(\lambda)\,d\lambda = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda kT}}-1}\,|d\lambda|$$

Since $c = \lambda f$, we can substitute in for $f{=}c\lambda^{-1}$ in the first equation to derive the second equation. We also need to substitute $df = -c\lambda^{-2}d\lambda$.

$$\begin{aligned}
\tilde{u}(f)\,df &= \frac{8\pi f^2}{c^3}\frac{hf}{e^{\frac{hf}{kT}}-1}\,|df| \\
&= \frac{8\pi\left(c\lambda^{-1}\right)^2}{c^3}\frac{hc\lambda^{-1}}{e^{\frac{hc\lambda^{-1}}{kT}}-1}\left|-c\lambda^{-2}d\lambda\right| \\
&= \frac{8\pi c^2}{\lambda^2 c^3}\frac{hc}{\lambda e^{\frac{hc\lambda^{-1}}{kT}}-1}\frac{c}{\lambda^2}\,|d\lambda| \\
&= \frac{8\pi}{\lambda^5}\frac{h}{e^{\frac{hc}{\lambda kT}}-1}c\,|d\lambda| \\
&= \frac{8\pi hc}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda kT}}-1}\,|d\lambda|
\end{aligned}$$

Since this is the same as the right hand side of the second equation, we can conclude that $\tilde{u}(f)\,df = u(\lambda)\,d\lambda$.

Problem 5

Integrate Plank's law over all wavelengths to derive Steffan's law: that the total power radiated by a black body is

$$R = \sigma T^4$$

where

$$\sigma = \frac{2\pi^5 k^4}{15h^3c^2}$$

using

$$\int_0^\infty \frac{x^3\,dx}{e^x-1} = \frac{\pi^4}{15}$$

Plank's law can be written in two ways. The first is:

$$u(\lambda) = \frac{8\pi hc\lambda^{-5}}{e^{hc/\lambda kT}-1}$$

The above form is a spectral energy density, which has units of energy per unit volume per unit frequency. Another way to write Plank's law is:

$$I(\lambda) = \frac{2\pi hc^2\lambda^{-5}}{e^{hc/\lambda kT}-1}$$

This has units of power per unit area per solid angle (whatever that means) per unit frequency. Since R is a power per unit area, the $I(\lambda)$ is the form of Plank's law that we want to use. If we use $u(\lambda)$, the resulting equation will be different by a factor of $\frac{4}{c}$.

We can integrate this, using $x = \frac{hc}{\lambda kT}$, $dx = -\frac{hc}{\lambda^2 kT}d\lambda$, $\lambda = \frac{hc}{xkT}$, and $\frac{d\lambda}{\lambda^2} = -\frac{kT}{hc}dx$:

$$\begin{aligned}
\int_0^\infty u(\lambda)\,d\lambda &= \int_0^\infty \frac{2\pi hc^2\lambda^{-5}}{e^{hc/\lambda kT}-1}d\lambda \\
&= \int_0^\infty \frac{2\pi hc^2\lambda^{-3}}{e^{hc/\lambda kT}-1}\left(\frac{d\lambda}{\lambda^2}\right) \\
&= \int_\infty^0 \frac{2\pi hc^2\left(\frac{hc}{xkT}\right)^{-3}}{e^x-1}\left(-\frac{kT}{hc}dx\right) \\
&= -\int_0^\infty \frac{2\pi hc^2k^3T^3x^3}{h^3c^3e^x-1}\left(-\frac{kT}{hc}dx\right) \\
&= \int_0^\infty \frac{2\pi k^4T^4x^3}{h^3c^2e^x-1}dx \\
&= \frac{2\pi k^4T^4}{h^3c^2}\int_0^\infty \frac{x^3}{e^x-1}dx \\
&= \frac{2\pi k^4T^4}{h^3c^2}\frac{\pi^4}{15} \\
&= \frac{2\pi^5k^4T^4}{15h^3c^2} \\
&= \left(\frac{2\pi^5k^4}{15h^3c^2}\right)T^4 \\
&= \sigma T^4
\end{aligned}$$

Problem 6

Set the derivative of Plank's law equal to zero to show Wien's displacement law: that

$$\lambda_m T = \text{const} = 2.989 \times 10^{-3} \text{m} \cdot \text{K}$$

where λ_m is the wavelength of maximum intensity radiation from a blackbody. The Sun's surface temperature is 5800 K. Calculate λ_m. What color is the sun?

Plank's law is:

$$u(\lambda) = \frac{8\pi hc\lambda^{-5}}{e^{hc/\lambda kT} - 1}$$

Differentiating, we get:

$$\begin{aligned} \frac{du}{d\lambda} &= 8\pi hc\left(-\frac{5\lambda^{-6}}{e^{hc/\lambda kT}-1}\right) \\ &\quad +8\pi hc\left(-\frac{\lambda^{-5}}{\left(e^{hc/\lambda kT}-1\right)^2}e^{hc/\lambda kT}\left(-\frac{hc}{\lambda^2 kT}\right)\right) \\ &= 8\pi hc\left(-\frac{5\lambda^{-6}}{e^{hc/\lambda kT}-1}+\frac{hce^{hc/\lambda kT}}{\lambda^7 kT\left(e^{hc/\lambda kT}-1\right)^2}\right) \end{aligned}$$

Setting this equal to zero, we get:

$$\begin{aligned} 0 &= 8\pi hc\left(-\frac{5\lambda^{-6}}{e^{hc/\lambda kT}-1}+\frac{hce^{hc/\lambda kT}}{\lambda^7 kT\left(e^{hc/\lambda kT}-1\right)^2}\right) \\ 0 &= -\frac{5\lambda^{-6}}{e^{hc/\lambda kT}-1}+\frac{hce^{hc/\lambda kT}}{\lambda^7 kT\left(e^{hc/\lambda kT}-1\right)^2} \\ 0 &= \frac{1}{\lambda^6\left(e^{hc/\lambda kT}-1\right)}\left(-5+\frac{hce^{hc/\lambda kT}}{\lambda kT\left(e^{hc/\lambda kT}-1\right)}\right) \\ 0 &= -5+\frac{hce^{hc/\lambda kT}}{\lambda kT\left(e^{hc/\lambda kT}-1\right)} \end{aligned}$$

Now we can substitute $x = \frac{hc}{\lambda kT}$ into the equation, to get:

$$0 = \frac{xe^x}{e^x - 1} - 5$$

This (Wikipedia) "cannot be solved in terms of elementary functions. It can be solved in terms of Lambert's Product Log function but an exact solution is not important in this derivation." However, we can estimate x fairly easily. First, let's rearrange the function so that it reads:

$$\begin{aligned} 5\left(e^x - 1\right) &= xe^x \\ x &= 5\left(1 - e^{-x}\right) \end{aligned}$$

Now we can let $e^{-x} = 0$, and estimate $x \approx 5$. Substituting that back into the right side of the equation, we have $x \approx 5\left(1 - e^{-5}\right) \approx 4.96631027$. Substituting this value back into the equation, we get $x \approx 5\left(1 - e^{-4.96631027}\right) \approx 4.96515593$. Substituting that back into the equation one last time, we get $x \approx 5\left(1 - e^{-4.96515593}\right) \approx 4.96511569$. Between the last two iterations, the value of x changed by less than 0.00005. This is a reasonable approximation. So, we can say that $x \approx 4.965$

Now if we substitute this back into $x = \frac{hc}{\lambda kT}$, we get, with a little shuffling around:

$$\begin{aligned} \lambda T &= \frac{hc}{kx} \\ &\approx \frac{\left(6.63 \times 10^{-34}\frac{\text{m}^2\text{kg}}{\text{s}}\right)\left(3.00 \times 10^8\frac{\text{m}}{\text{s}}\right)}{\left(1.381 \times 10^{-23}\frac{\text{kg}\cdot\text{m}^2}{\text{s}^2\text{K}}\right)(4.965)} \\ &\approx 2.90 \times 10^{-3}\text{m} \cdot \text{K} \end{aligned}$$

Approximation errors aside, this agrees well with the the problem statement. The color of the sun, if it is at 5800 K, is

$$\lambda \approx \frac{2.989 \times 10^{-3}}{5800} \approx 515 \text{ nm}$$

This makes the sun's color of highest contribution green, close to blue. However, the sun's color is still white. For details, please see http://en.wikipedia.org/wiki/Planckian_locus