

Problem 1

In 1866 when atoms and molecules were still quite hypothetical, Joseph Loschmidt used kinetic energy theory to get the first reasonable estimate of molecular size. He used the liquid to gas expansion ratio of air $\epsilon = n_g/n_l$ (where $n = N/V$) together with the mean free path λ between collisions to calculate the average diameter s of air molecules. In the following diagram, σ is the cross sectional area of interaction between two air molecules (cross section). The purpose of this problem is to reproduce his results and show the profound consequences in atomic physics.

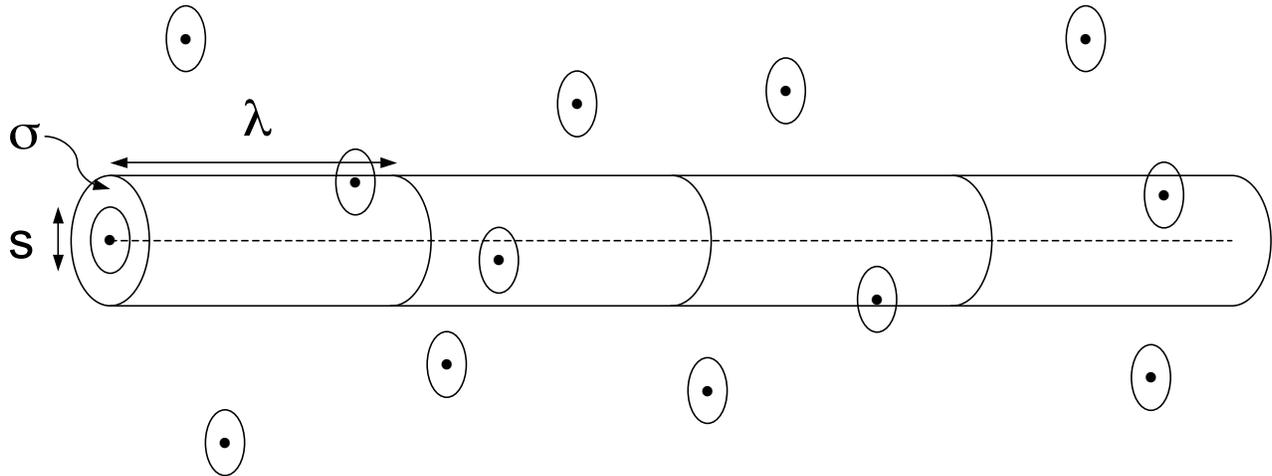


Figure 1: Volume of Interaction for Air Molecules

Part a

Justify that the number density of air [molecules/cm³] is $n_g = 1/\lambda\sigma$.

Since σ is the cross-section of interaction, that is, the radius of σ gives the maximum distance from the probe that the center of an air molecule can be and still interact with the probe, and λ is the average distance that the probe can travel without encountering an air molecule, $\lambda\sigma$ defines a volume in which only one air molecule, on average, exists. Since there is only one molecule in this volume, the number density can be expressed as one molecule per unit volume, or

$$n_g = \frac{1 \text{ air molecule}}{\lambda\sigma}$$

Part b

Justify that $\sigma = \pi s^2$, not, for example, $\pi (s/2)^2$.

We use s here, the diameter of the probe, instead of, say, $s/2$ (the radius) because we want the total area of interaction. Since both the probe and the air molecule occupy a volume, we cannot conclude that the cross sectional area of the probe is sufficient. Instead, we need to consider that the very outer shell of both the probe and the air molecule can interact. So, the area of interaction must have a radius that is equal to the maximum distance between the centers of the probe and the molecule when the two are still interacting. Since both the probe and the air molecules can be considered to be spheres, the distance from the center of the probe or air molecule to its outer shell is equal to the radius of the probe or air molecule, and the total maximum distance between the probe and the air molecule is, then, equal to the sum of the radii of the probe and the air molecule. If we assume that the probe and the air molecule are about the same size, with diameter s , this becomes $2(s/2)$, or just s .

Part c

Show that $\epsilon = s/6\lambda$, assuming that the molecules are tightly packed in a liquid. (Extra credit) Explain why Loschmidt used $\epsilon = s/8\lambda$.

Since $\epsilon = n_g/n_l$, and each $n = N/V$, we can estimate the condensation coefficient by using $n_g = 1/\lambda\sigma$ (the molecule number density of an individual molecule in a gas from part (a)) and $n_l = \frac{1}{\frac{4}{3}\pi(s/2)^3}$, which is the molecule number density of any individual spherical gas molecule in a liquid. This gives:

$$\begin{aligned}\epsilon &= \frac{\frac{1}{\lambda\sigma}}{\frac{1}{\frac{4}{3}\pi(s/2)^3}} \\ &= \frac{\frac{4}{3}\pi \frac{s^3}{8}}{\lambda\pi s^2} \\ &= \frac{\frac{1}{3}s \frac{1}{2}}{\lambda} \\ \epsilon &= \frac{s}{6\lambda}\end{aligned}$$

Instead of $\epsilon = s/6\lambda$, Loschmidt used $\epsilon = s/8\lambda$. Examining “On the Size of the Air Molecules” by Loschmidt [<http://www.chemteam.info/Chem-History/Loschmidt-1865.html>] and “On the Dynamical Theory of Gases” by Clausius [http://books.google.com/books?id=6ebK5yP1q1IC&pg=PA434&lpg=PA434&dq=Clausius+%22On+the+dynamical+theory+of+gases%22&source=bl&ots=-n-YD_SQpR&sig=db3QujaX9AXYqNQ9POyEwQEpmPY&hl=en&ei=f7ZrS7fxI43V8AbM4uiPBg&sa=X&oi=book_result&ct=result&resnum=4&ved=0CBgQ6AEwAw#v=onepage&q=Clausius%20%22On%20the%20dynamical%20theory%20of%20gases%22&f=false], the reason for this disparity obviously arises because Loschmidt and Clausius did not assume that the air molecules in a sample would be stationary, and that they would be moving at the same velocity as the probe. Clausius asserted that this increases the number of collisions by a factor of $\frac{4}{3}$. In “On the Dynamical Theory of Gasses”, Clausius attempted to prove this assertion.

As a starting point, he the same simplifying assumption that we made, that the probe was in motion but the other air molecules were not. This led to the same conclusion that we reached, that is, that:

$$\frac{1}{n_g} = \lambda\pi s^2$$

Actually, he was more interested in the number of collisions in a given time:

$$N_s = v\pi s^2 N$$

Where N_s is the number of collisions for this stationary system, v is the “absolute” velocity of the probe, N is the number of stationary molecules in the unit space, and s is, as before, the maximum distance between the centers of two interacting molecules. Next, he removes the simplifying assumption, so that all molecules can have motion, and he replaces v with the relative velocity between the probe and the mean of the velocities of the other air molecules, r . Not surprisingly, this changes the number of collisions to:

$$N_d = r\pi s^2 N$$

The ratio, then, is $N_s/N_d = v/r$. Clausius notes here that Maxwell arrived at the same conclusion independently. However, they apparently differed on the determination of the mean value, r . Quoting the letter directly (or almost... ϕ has been used in place of another character in a different script):

Let u be the velocity of any molecule m , and ϕ the angle between the direction of its motion and that of the molecule μ ; the relative velocity between μ and m will then be

$$\sqrt{u^2 + v^2 - 2uv \cos \phi}$$

When the molecules, $m, m_1, m_2 \dots$ all move with the same velocity, in other words, when u is constant and ϕ alone variable from one molecule to another, the mean value can be easily calculated.

Maxwell apparently believed that, in this case, $r = \sqrt{u^2 + v^2} = v\sqrt{2}$. However, Clausius disagrees:

Since all directions are equally probable for the molecules $m, m_1, m_2 \dots$, the number of those whose lines of motions make angles between ϕ and $\phi + d\phi$ with the line in which μ moves will have to the whole number of molecules the same ratio that a spherical zone with the polar angle ϕ and the breadth $d\phi$ has to the whole surface of the sphere, in other words, the ratio

$$2\pi \sin \phi d\phi : 4\pi$$

The number of such molecules in the unit of volume is consequently

$$N \cdot \frac{1}{2} \sin \phi d\phi$$

In order to obtain the required mean value r , the last expression must be multiplied by the relative velocity which corresponds to it, the product integrated between the limits 0 and π , and the integral divided by N . Hence

$$r = \frac{1}{2} \int_0^\pi \sqrt{u^2 + v^2 - 2uv \cos \phi} \cdot \sin \phi d\phi$$

This gives at once

$$r = \frac{1}{6uv} \left[(u^2 + v^2 + 2uv)^{\frac{3}{2}} - (u^2 + v^2 - 2uv)^{\frac{3}{2}} \right]$$

whence we may deduce

$$r = v + \frac{1}{3} \frac{u^2}{v}, \text{ when } u < v$$

and

$$r = u + \frac{1}{3} \frac{v^2}{u}, \text{ when } u > v$$

When $u = v$, both results coincide in value with

$$r = \frac{4}{3}v$$

and thus verify my assertion.

This changes the estimate of n_g , which, through Clausius' work, becomes:

$$n_g = \frac{3}{4\pi\lambda s^2} = \frac{3}{4\lambda\sigma}$$

Using $n_l = \frac{6}{\pi s^3}$, we can show that:

$$\begin{aligned} \epsilon &= \frac{n_g}{n_l} = \frac{\frac{3}{4\lambda\sigma}}{\frac{6}{\pi s^3}} \\ &= \left(\frac{3}{4\lambda\pi s^2} \right) \left(\frac{\pi s^3}{6} \right) \\ &= \left(\frac{1}{4\lambda} \right) \left(\frac{s}{2} \right) \\ &= \frac{s}{8\lambda} \end{aligned}$$

This is why Loschmidt used $\epsilon = s/8\lambda$.

Part d

Given the density of liquid air $\rho_l = 0.870 \text{ g/cm}^3$, the molar volume of an ideal gas $V_m = 24 \text{ L/mol}$, and the atomic weight of air $A = 35.6 \text{ g/mol}$, calculate the expansion ratio ϵ from liquid to air at STP (1 atm, 20°C.) Loschmidt had to estimate ρ_l , since air had not been liquefied by 1866.

Here, we need to find $\epsilon = n_g/n_l$. To determine this ratio, it is important to calculate n_g and n_l in the same units. n_g can be found easily by:

$$n_g = \frac{N_A}{V_m}$$

To find n_l :

$$n_l = \frac{\rho_l N_A}{A}$$

For ϵ this gives us:

$$\begin{aligned} \epsilon &= \frac{n_g}{n_l} \\ &= \frac{\frac{N_A}{V_m}}{\frac{\rho_l N_A}{A}} \\ &= \frac{A}{V_m \rho_l} \\ &= \frac{35.6 \text{ g/mol}}{(0.870 \text{ g/cm}^3)(24 \text{ L/mol})(1000 \text{ cm}^3/\text{L})} \\ &= 0.00170 \end{aligned}$$

This is on the same order as the condensation coefficients calculated by Loschmidt for various gasses, though it is quite a bit larger than his calculated condensation coefficient for air itself, which was 0.000866.

Part e

Maxwell determined $\lambda = 62 \text{ nm}$ from viscosity measurements of air. A later extraction $\lambda = 140 \text{ nm}$ by Oskar Emil Meyer (don't laugh!) was less accurate. Use this to calculate s and σ .

Using our equation for $\epsilon = s/6\lambda$, we can easily determine s . σ can then be determined using $\sigma = \pi s^2$. For $\lambda = 62 \text{ nm}$:

$$\begin{aligned} s &= 6\lambda\epsilon \\ &= 6(62 \text{ nm})(0.00170) \\ &= 0.6324 \text{ nm} \end{aligned}$$

And:

$$\begin{aligned} \sigma &= \pi(0.6324)^2 \\ &= 1.26 \text{ nm}^2 \end{aligned}$$

For $\lambda = 140 \text{ nm}$:

$$\begin{aligned} s &= 6\lambda\epsilon \\ &= 6(140 \text{ nm})(0.00170) \\ &= 1.428 \text{ nm} \end{aligned}$$

And:

$$\begin{aligned} \sigma &= \pi(1.428)^2 \\ &= 6.406 \text{ nm}^2 \end{aligned}$$

Part f

Using σ , calculate the density of air n_g [cm^{-3}] at STP (the Loschmidt number or 1 Amagat).

Using σ for $\lambda = 62 \text{ nm}$ from part (e), we have:

$$\begin{aligned} n_g &= \frac{1}{\sigma\lambda} \\ &= \frac{1}{(1.26 \text{ nm}^2)(62 \text{ nm})} \\ &= 1.28 \times 10^{19} \text{ cm}^{-3} \end{aligned}$$

Part g

Using n_g and V_m calculate Avogadro's number N_A (also called the Loschmidt number).

Given $V_m = 24 \text{ L/mol}$ and $n_g = 1.28 \times 10^{19} \text{ cm}^{-3}$, we have:

$$\begin{aligned} N_A &= V_m n_g \\ &= \left(\frac{24 \text{ L}}{\text{mol}} \right) \left(\frac{1.28 \times 10^{19} \text{ molecules}}{\text{cm}^3} \right) \\ &= 3.072 \times 10^{23} \text{ molecules/mol} \end{aligned}$$

Amazingly, this is on the same order of magnitude as the actual Avogadro's number, 6.022×10^{23} molecules/mol.

Part h

Faraday's constant $F = N_A e = 96485 \text{ C/mol}$ was determined by weighing the amount of silver deposited during electrolysis of silver ions. Use this to calculate the elementary charge e .

Easily, we have:

$$\begin{aligned} e &= \frac{F}{N_A} \\ &= \frac{96485 \text{ C/mol}}{3.072 \times 10^{23} \text{ molecules/mol}} \\ &= 3.141 \times 10^{-19} \text{ C/molecule} \end{aligned}$$

Once again, very close to the actual value, $1.602 \times 10^{-19} \text{ C}$.

Part i

Using the cathode ray tube, Thompson discovered electrons, negatively charged particles with charge-to-mass ratio $e/m = 1.75 \times 10^8 \text{ C/g}$. Calculate the mass of an electron.

Here, we have;

$$\begin{aligned} m &= \frac{e}{1.75 \times 10^8 \text{ C/g}} \\ &= \frac{3.141 \times 10^{-19} \text{ C}}{1.75 \times 10^8 \text{ C/g}} \\ &= 1.795 \times 10^{-27} \text{ g} \\ &= 1.795 \times 10^{-30} \text{ kg} \end{aligned}$$

Again, very close to the actual mass of an electron, $9.109 \times 10^{-31} \text{ kg}$.

Part j

Using the atomic weight $A = 1.008$ g/mol of hydrogen, calculate the mass of a proton.

Since a hydrogen atom is made up of one proton and one electron, the mass of the proton is:

$$\begin{aligned} m_p &= m_a - m_e \\ &= \frac{1.008 \text{ g/mol}}{3.072 \times 10^{23} \text{ molecules/mol}} \\ &\quad - 1.795 \times 10^{-27} \text{ g} \\ &= 3.279 \times 10^{-24} \text{ g} \end{aligned}$$

Of course, subtracting the electron makes very little difference, since it is so much smaller than the mass of the proton.

Problem 2

Consider the scattering of hard spheres as discussed in class. It is assumed the target is infinitely massive so it does not recoil, and the probe scatters by reflection, i.e. the angle of reflection equals the angle of incidence. The radius of the probe and target are a and A respectively.

Part a

Determine the relation between the impact parameter b and the scattering angle θ .

Since we're scattering hard spheres here, we can assume that the two spheres will never "intersect". Also, the problem definition states that the scattering will be through reflection, which completely defines the behavior of the two interacting objects.

At the point at which the probe impacts the target, the probe will reflect off the target at some angle ϕ that equals the angle of incidence. This angle is defined with respect to a line through both the radius of the target and the point of impact.

The impact parameter for an individual event, b , can now be defined in terms of the distance of the center of the probe from the center of the cross section as the sum of the portion of the radii of the probe and target that are in the radial direction of the cross section, or $b = A \sin(\phi) + a \sin(\phi)$, which leaves us with:

$$b = (A + a) \sin \phi$$

However, we ultimately want to know b in terms of θ , the scattering angle, not ϕ . Fortunately, since we know that the probe reflects off the target, we know that the angle that the reflected ray makes with the angle of incidence is equal to the angle that the impact ray makes with the angle of incidence, and then the impact parameter $\theta = 180 - 2\phi$. So, $\phi = \frac{180 - \theta}{2}$. This gives us

$$\begin{aligned} b &= (A + a) \sin \left(\frac{180 - \theta}{2} \right) \\ &= (A + a) \sin \left(90 - \frac{\theta}{2} \right) \\ &= (A + a) \cos \left(\frac{\theta}{2} \right) \end{aligned}$$

This gives the correct result at $\theta = 0^\circ$ as $b = A + a$, which is the expected maximum impact parameter, and gives $b = 0$ when $\theta = 180^\circ$, which is once again correct, since a head-on collision should result in a rebound of the probe.

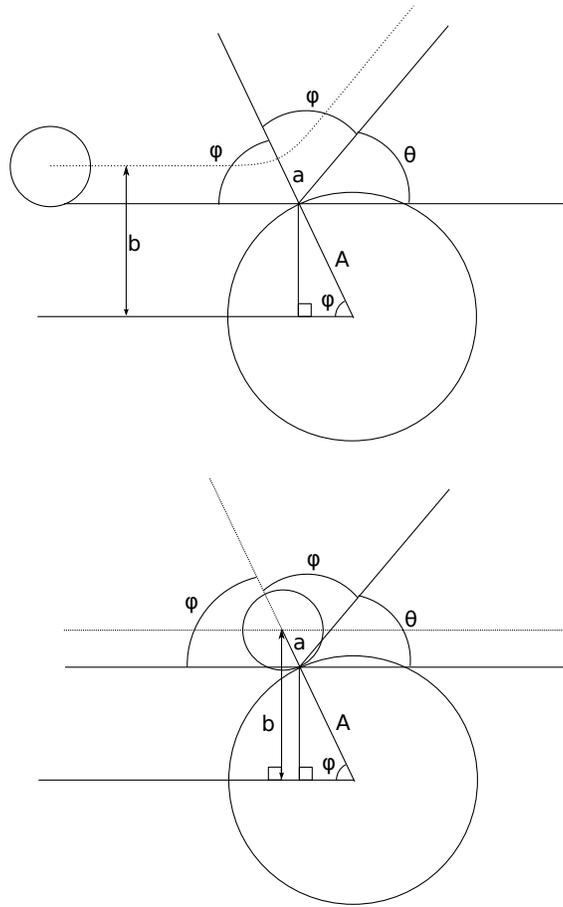


Figure 2: Hard Interaction between Spheres

Part b

Using part (a), calculate the cross section $d\sigma/d\Omega$. Plot it as a function of θ .

To calculate $\frac{d\sigma}{d\Omega}$, we can first find $d\sigma$ and $d\Omega$ separately. Since $\sigma = \pi b^2$, $d\sigma = 2\pi b db$. Ω is defined to be the solid angle, or $\Omega = 2\pi(1 - \cos\theta)$. So, $d\Omega = 2\pi \sin\theta d\theta$. This θ is the same scattering angle as in part (a). This gives us:

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{2\pi b db}{2\pi \sin\theta d\theta} \\ &= \frac{b db}{\sin\theta d\theta} \end{aligned}$$

We can also substitute in for b and db by using $b = (A + a) \cos\theta/2$ and $db = -\frac{1}{2}(A + a) \sin\theta/2 d\theta$. However, since $d\sigma$ represents an area and $d\Omega$ represents a volume, $\frac{d\sigma}{d\Omega}$ must be positive, and making these substitutions will not give us a positive $\frac{d\sigma}{d\Omega}$. However, $\frac{b db}{\sin\theta d\theta}$ can be thought of as a Jacobian that is used to change coordinates for a volumetric differential, and those are generally taken to be positive. So, using $\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta} \left| \frac{db}{d\theta} \right|$ this gives us:

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{\left((A + a) \cos\frac{\theta}{2} \right)}{\sin\theta} \left| \frac{\left(-\frac{1}{2} (A + a) \sin\frac{\theta}{2} \right) d\theta}{d\theta} \right| \\ \frac{d\sigma}{d\Omega} &= \frac{\left((A + a) \cos\frac{\theta}{2} \right) \left(\frac{1}{2} (A + a) \sin\frac{\theta}{2} \right) d\theta}{\sin\theta d\theta} \\ &= \frac{\frac{1}{2} (A + a)^2 2 \cos\frac{\theta}{2} \sin\frac{\theta}{2} d\theta}{2 \sin\theta d\theta} \end{aligned}$$

Using $\sin 2u = 2 \sin u \cos u$:

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{\frac{1}{2}(A+a)^2 \sin \theta d\theta}{2 \sin \theta d\theta} \\ &= \frac{(A+a)^2}{4}\end{aligned}$$

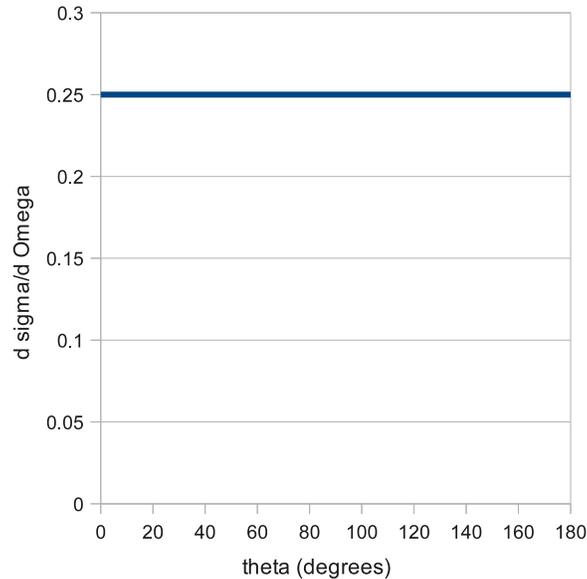


Figure 3: $\frac{d\sigma}{d\Omega}$ In Terms of θ for $A + a = 1$

Part c

Integrate part (b) to calculate the total cross section

$$\begin{aligned}\sigma_{tot} &= \int d\sigma = \int_{4\pi} \frac{d\sigma}{d\Omega} d\Omega \\ &= \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{d\sigma}{d\Omega} \sin \theta d\theta d\phi\end{aligned}$$

What is an intuitive interpretation of the result?

Substituting and integrating, we get:

$$\begin{aligned}\sigma_{tot} &= \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{d\sigma}{d\Omega} \sin \theta d\theta d\phi \\ &= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{(A+a)^2}{4} \sin \theta d\theta d\phi \\ &= 2\pi \int_{\theta=0}^{\pi} \frac{(A+a)^2}{4} \sin \theta d\theta \\ &= \frac{(A+a)^2}{2} \pi \int_{\theta=0}^{\pi} \sin \theta d\theta \\ &= \frac{(A+a)^2}{2} \pi [-\cos \theta]_0^{\pi} \\ &= \frac{(A+a)^2}{2} \pi (1 + 1) \\ &= (A+a)^2 \pi\end{aligned}$$

This can be interpreted as the maximum cross-sectional area of the incident particle path corridor of interaction with the target. That is, this is the maximum cross section, given target and probe radii of A and a , respectively.

Part d

In a model of alpha particle scattering from gold nuclei, let $a = r_\alpha = 1.7$ fm and $A = r_{\text{Au}} = 7.3$ fm. What value of b leads to scattering at $\theta = 90^\circ$? 10×10 cm² detectors are placed 2 m away from the target, which is a 0.00004 cm thick gold foil, $Z = 79$, $A = 197$ g/mol, $\rho = 19.3$ g/cm³. What percentage of the incident α particles will scatter into detectors at $\theta = 15^\circ$ and $\theta = 165^\circ$?

Assuming that we are using the same scattering model as specified in the first parts of this problem, we have $b = (A + a) \cos \frac{\theta}{2}$, and $b = (9 \text{ fm}) \cos \frac{\pi/2}{2} = 6.36$ fm leads to scattering at $\theta = 90^\circ$.

To find the percentage of incident α particles, we can use $\frac{d\sigma}{d\Omega} = \frac{dN}{I_0 n t A_{sc}} \frac{r^2}{A_{sc}}$, where the fraction is $f = \frac{dN}{I_0}$, n is the number of particles per unit volume in the target, t is the thickness of the target, A_{sc} is the area of the detector, and r is the distance between the target and the detector. Solving for f , we get:

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{dN}{I_0 n t A_{sc}} \frac{r^2}{A_{sc}} \\ \frac{d\sigma}{d\Omega} &= f \frac{r^2}{n t A_{sc}} \\ \frac{d\sigma}{d\Omega} \frac{n t A_{sc}}{r^2} &= f \end{aligned}$$

Since we know from part b that $\frac{d\sigma}{d\Omega} = \frac{(A+a)^2}{4}$, this becomes:

$$\frac{(A+a)^2}{4} \frac{n t A_{sc}}{r^2} = f$$

This apparently is independent of the scattering angle, so f is the same regardless of θ . We can calculate n , the number of target atoms per unit volume, by renaming $A = M = 197$ g/mol, to avoid a name conflict between the various A 's, and calculating:

$$\begin{aligned} n &= \frac{\rho N_A}{M} \\ &= \frac{(19.3 \text{ g/cm}^3) (6.022 \times 10^{23} \text{ atoms/mol})}{197 \text{ g/mol}} \\ &= 5.90 \times 10^{22} \text{ atoms/cm}^3 \end{aligned}$$

Now, we can substitute in to determine f (making sure to convert the various length measurements appropriately):

$$\begin{aligned} f &= \frac{(A+a)^2}{4} \frac{n t A_{sc}}{r^2} \\ &= \frac{(7.3 \text{ fm} + 1.7 \text{ fm})^2}{4} \frac{(5.90 \times 10^{22} \text{ atoms/cm}^3) (0.00004 \text{ cm}) (100 \text{ cm}^2)}{(2 \text{ m})^2} \\ &= 1.19475 \times 10^{-9} \end{aligned}$$

Problem 3

We will now compare the above with Rutherford scattering (see Beiser, Chapter 4).

Also, the coulomb force is $F = \frac{2Z_2k\epsilon^2}{r^2}\hat{r}$ where \hat{r} is the radian unit vector, and $F_{\Delta P} = F \cos \phi$, where ϕ is the angular location of the projective relative to the symmetry plane. This gives:

$$\Delta P = 2Zke^2 \int \frac{\cos \phi}{r^2} dt = 2mv_0 \sin \frac{\theta}{2}$$

We can use conservation of angular momentum to help evaluate the impulse integral:

$$L = mv_0b = mr^2 \frac{d\phi}{dt}$$

$$r^2 = \frac{v_0b}{d\phi/dt}$$

Substitution gives:

$$2mv_0 \sin \frac{\theta}{2} = 2Zke^2 \int \frac{\cos \phi}{v_0b} \frac{d\phi}{dt} dt$$

$$= \frac{2Zke^2}{v_0b} \int_{\phi_i}^{\phi_f} \cos \phi d\phi$$

Extending the entrance and exit paths of the scattering trajectory to infinity gives the limits of the angle:

$$\phi_i = -\frac{1}{2}(\pi - \theta)$$

$$\phi_f = \frac{1}{2}(\pi - \theta)$$

Using the angle difference identity:

$$\int_{\phi_i}^{\phi_f} \cos \phi d\phi = \sin \left[\frac{1}{2}(\pi - \theta) \right] - \sin \left[-\frac{1}{2}(\pi - \theta) \right]$$

$$= 2 \cos \frac{\theta}{2}$$

Solving for the impact parameter b:

$$2mv_0 \sin \frac{\theta}{2} = \frac{2Zke^2}{v_0b} 2 \cos \frac{\theta}{2}$$

$$b = \frac{\frac{2Zke^2}{v_0} 2 \cos \frac{\theta}{2}}{2mv_0 \sin \frac{\theta}{2}}$$

$$= \frac{Zke^2 \cos \frac{\theta}{2}}{mv_0^2 \sin \frac{\theta}{2}}$$

$$= \frac{2Zke^2}{mv_0^2} \cot \frac{\theta}{2}$$

Since $k = \frac{1}{4\pi\epsilon_0}$ and $mv_0^2 = 2E$, this gives us:

$$b = \frac{2Ze^2}{4\pi\epsilon_0 2E} \cot \frac{\theta}{2}$$

$$= \frac{Ze^2}{4\pi\epsilon_0 E} \cot \frac{\theta}{2}$$

Rearranging, we have:

$$\cot \frac{\theta}{2} = \frac{4\pi\epsilon_0 E}{Ze^2} b$$

Since $\cot \frac{90}{2} = \cot 45 = 1$, we have $b = \frac{Ze^2}{4\pi\epsilon_0 E}$ leading to scattering at 90° . This can be calculated to give us:

$$\begin{aligned} b &= \frac{Ze^2}{4\pi\epsilon_0 E} \\ &= \frac{79e^2}{4\pi\epsilon_0 (7.7 \text{ MeV})} \\ &= 14.77 \text{ fm} \end{aligned}$$

Part b

Calculate $d\sigma/d\Omega$ and plot it as a function of θ on the same graph as problem 1.

Once again, we have $d\sigma = 2\pi b db$ and $d\Omega = 2\pi \sin \theta d\theta$. We also have $b = \frac{Ze^2}{4\pi\epsilon_0 E} \cot \frac{\theta}{2}$ and $db = -\frac{Ze^2}{8\pi\epsilon_0 E} \csc^2 \frac{\theta}{2} d\theta$, but that $\frac{d\sigma}{d\Omega} = \frac{b}{\sin \theta} \left| \frac{db}{d\theta} \right|$. This gives us:

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{\left(\frac{Ze^2}{4\pi\epsilon_0 E} \cot \frac{\theta}{2} \right)}{\sin \theta} \left| \frac{\left(-\frac{Ze^2}{8\pi\epsilon_0 E} \csc^2 \frac{\theta}{2} d\theta \right)}{d\theta} \right| \\ &= \frac{\left(\frac{Ze^2}{4\pi\epsilon_0 E} \right)^2 \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} \left(\frac{1}{2} \right) \left(\frac{1}{\sin^2 \frac{\theta}{2}} \right)}{\sin \theta} \\ &= \frac{\left(\frac{Ze^2}{4\pi\epsilon_0 E} \right)^2 \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} \left(\frac{1}{2} \right) \left(\frac{1}{\sin^2 \frac{\theta}{2}} \right)}{\sin \theta} \end{aligned}$$

Using $\sin \left(2\frac{\theta}{2} \right) = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$:

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{\left(\frac{Ze^2}{4\pi\epsilon_0 E} \right)^2 \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} \left(\frac{1}{2} \right) \left(\frac{1}{\sin^2 \frac{\theta}{2}} \right)}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} \\ \frac{d\sigma}{d\Omega} &= \frac{\left(\frac{Ze^2}{4\pi\epsilon_0 E} \right)^2}{4 \sin^4 \frac{\theta}{2}} \end{aligned}$$

If we assume that $\left(\frac{Ze^2}{4\pi\epsilon_0 E} \right)^2 = 1 \text{ fm}^2$, which is not necessarily impossible, we can graph $\frac{d\sigma}{d\Omega} = \frac{1 \text{ fm}^2}{4 \sin^4 \frac{\theta}{2}}$, shown in Figure 5.

Part c

In the same conditions as in problem 1, what percentage of the incident α particles will scatter into the detector at i) $\theta = 15^\circ$, ii) $\theta = 165^\circ$?

We still have:

$$\begin{aligned} f &= \frac{d\sigma}{d\Omega} \frac{ntA_{sc}}{r^2} \\ &= \frac{\left(\frac{Ze^2}{4\pi\epsilon_0 E} \right)^2}{4 \sin^4 \frac{\theta}{2}} \frac{ntA_{sc}}{r^2} \end{aligned}$$

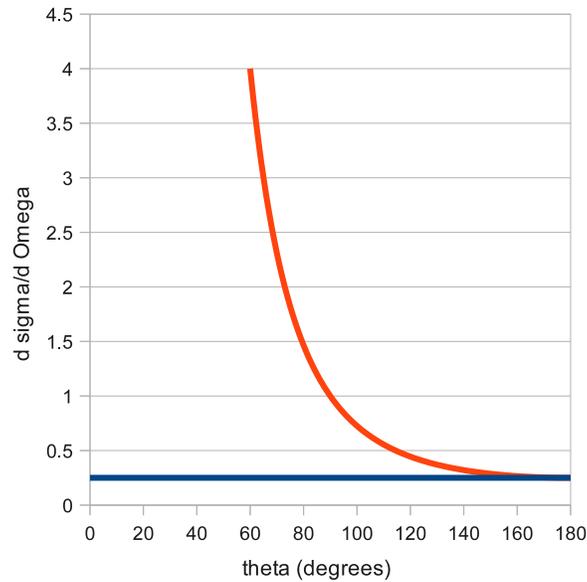


Figure 5: $\frac{d\sigma}{d\Omega}$ In Terms of θ for $A + a = 1$ and $\left(\frac{Ze^2}{4\pi\epsilon_0 E}\right)^2 = 1 \text{ fm}^2$

We know that that $E = 7.7 \text{ MeV}$ for the α particles, and we use $Z = 79$ for gold. Also, we can use $\frac{1}{4\pi\epsilon_0} = k_e$, and $k_e e^2 = 1.44 \text{ eV} \cdot \text{nm} = 1.44 \text{ MeV} \cdot \text{fm}$ to simplify the calculations later. We also know that $A_{xc} = 100 \text{ cm}^2$, $t = 0.00004 \text{ cm}$, and $r = 2 \text{ m}$. Also:

$$\begin{aligned} n &= \frac{\rho N_A}{M} \\ &= \frac{(19.3 \text{ g/cm}^3) (6.022 \times 10^{23} \text{ atoms/mol})}{197 \text{ g/mol}} \\ &= 5.90 \times 10^{22} \text{ atoms/cm}^3 \end{aligned}$$

Now, we can substitute in for f (again making sure to convert the various length measurements appropriately). For $\theta = 15^\circ$, we get:

$$\begin{aligned} f &= \frac{\left(\frac{Ze^2}{4\pi\epsilon_0 E}\right)^2}{4 \sin^4 \frac{\theta}{2}} \frac{ntA_{sc}}{r^2} \\ &= \frac{\left(\frac{(79)1.44 \text{ MeV}}{7.7 \text{ MeV}}\right)^2}{4 \sin^4 \frac{15}{2}} \frac{(5.90 \times 10^{22} \text{ atoms/cm}^3) (0.00004 \text{ cm}) (100 \text{ cm}^2)}{(2 \text{ m})^2} \\ &= 4.15869383 \times 10^{-9} \end{aligned}$$

For $\theta = 165^\circ$, we get:

$$\begin{aligned} f &= \frac{\left(\frac{Ze^2}{4\pi\epsilon_0 E}\right)^2}{4 \sin^4 \frac{\theta}{2}} \frac{ntA_{sc}}{r^2} \\ &= \frac{\left(\frac{(79)1.44 \text{ MeV}}{7.7 \text{ MeV}}\right)^2}{4 \sin^4 \frac{165}{2}} \frac{(5.90 \times 10^{22} \text{ atoms/cm}^3) (0.00004 \text{ cm}) (100 \text{ cm}^2)}{(2 \text{ m})^2} \\ &= 1.13250296 \times 10^{-8} \end{aligned}$$

Part d

Explain why the total cross section is actually infinite. Up until now, we have ignored the electrons of the gold atom. What is the effect on small angle (large b) scattering from the entire atom? (hint: what is the total charge?)

The total cross section is infinite because the electric force, while decreasing with distance, never reaches zero at any finite distance (in theory, although I wonder if force can be quantized. . .) The effect from the entire atom (including the electrons) on small angle scattering is extremely small. Since the atom has a net zero charge, the α particle will experience no electric force due to the atom before the particle has breached the electron shells. Once the α particle has breached the electron shells, the "uniform density" shell should, again, have no effect, since the total force felt due to the shell should be balanced at all points near the nucleus. However, the α particle **will** feel the force due to the charge on the nucleus once it has passed the electron orbitals.

Problem 4.11

If a particle is deflected by 0.01° in each collision, about how many collisions would be necessary to produce an rms deflection of 10° ? (Use the result from the one-dimensional random walk problem in statistics stating that the rms deflection equals the magnitude of the individual deflections times the square root of the number of deflections.) Compare this result with the number of atomic layers in a gold foil of thickness 10^{-6} m, assuming that the thickness of each atom is $0.1 \text{ nm} = 10^{-10} \text{ m}$.

So, $D_{rms} = |D_i| \sqrt{N}$. In our case $D_i = 0.01^\circ$, and $D_{rms} = 10^\circ$. This gives us:

$$\begin{aligned}\sqrt{N} &= \frac{10^\circ}{0.01^\circ} = 1000 \\ N &= 1000^2 = 10^6\end{aligned}$$

Assuming a regular arrangement of atoms across the thickness of the foil, we can estimate the number of atoms across the thickness of the gold foil as $(10^{-6} \text{ m}) (1 \text{ atom}/10^{-10} \text{ m}) = 10^4$. This is two full orders of magnitude smaller than 10^6 . This number of collisions would produce $D_{rms} = 0.01^\circ \sqrt{10^4} = 1^\circ$.

Problem 4.16

If the angular momentum of Earth in its motion around the Sun were quantized like a hydrogen electron according to Equation 4-17, what would Earth's quantum number be? How much energy would be released in a transition to the next lowest level? Would that energy release (presumably as a gravity wave) be detectable? What would be the radius of that orbit? (The radius of Earth's orbit is $1.50 \times 10^{11} \text{ m}$.)

Equation 4-17 give us the equation for angular momentum as $L = mvr = n\hbar$. The mass of Earth is $5.9736 \times 10^{24} \text{ kg}$, and the velocity of the earth is, on average, $2.9783 \times 10^4 \text{ m/s}$. The quantum number at $r = 1.50 \times 10^{11} \text{ m}$, then, is:

$$\begin{aligned}n &= \frac{mvr}{\hbar} \\ &\approx \frac{(5.97 \times 10^{24} \text{ kg}) (1.5 \times 10^{11} \text{ m}) (2.98 \times 10^4 \frac{\text{m}}{\text{s}})}{1.05457148 \times 10^{-34} \text{ m}^2 \text{ kg/s}} \\ &\approx 2.53 \times 10^{74}\end{aligned}$$

To find the energy released by transitioning to the next lower n , we must know the energy at each orbital. We can use the equation derivations from problem 4.56 here, which gives $E_n = -\frac{G^2 M^2 m^3}{2n^2 \hbar^2}$. In this case, we can call $E_0 = \frac{G^2 M^2 m^3}{2\hbar^2}$,

since all of these are constants in this problem. The energy released, then, is

$$\begin{aligned} E &= E_{n+1} - E_n \\ &= -\frac{E_0}{(n+1)^2} + \frac{E_0}{n^2} \\ &= E_0 \left(\frac{1}{n^2} - \frac{1}{(n+1)^2} \right) \end{aligned}$$

This quantity, due to the size of the values involved, is close to impossible to calculate. Instead, we can normalize this quantity, so that:

$$\begin{aligned} n^2 (n+1)^2 E &= E_0 \left((n+1)^2 - n^2 \right) \\ n^2 (n+1)^2 E &= E_0 (n^2 + 2n + 1 - n^2) \\ n^2 (n+1)^2 E &= E_0 (2n + 1) \end{aligned}$$

Once again, this is essentially impossible to calculate, but we can make a reasonable estimate by using exponents. Since $E_0 = \frac{G^2 M^2 m^3}{2\hbar^2} = 1.05288033 \times 10^{201} \text{eV}$, we can estimate:

$$\begin{aligned} (10^{74})^2 ((10^{74}) + 1)^2 E &= (10^{201}) (2(10^{74}) + 1) \\ (10^{148}) (10^{148}) E &= (10^{201}) (10^{74}) \\ (10^{296}) E &= 10^{275} \\ E &= \frac{10^{275}}{10^{296}} \\ E &= 10^{-21} \text{eV} \end{aligned}$$

This is an extremely small amount of energy, on the order of 10^{21} times smaller than the energy released by an electron transitioning to the next lowest orbit. Since this amount of energy is not enough to excite any known energy detection device, it is unlikely that this transition could be detected using any known equipment. Any change in the radius of the Earth's orbit in this case would be basically imperceptible, so the final radius would still be about $1.5 \times 10^{11} \text{m}$.

Problem 4.41

Part a

The current i due to a charge q moving in a circle with frequency f_{rev} is qf_{rev} . Find the current due to the electron in the first Bohr orbit.

Since $q = e$ in this case, since the moving charge is the electron, $i = ef_{\text{rev}}$. The question is, then, what is the frequency of an electron in the first Bohr orbit? The radius of the first Bohr orbit is $r_1 = \frac{a_0}{Z}$. The velocity of the first Bohr orbital is $v_1 = Z\alpha c$. Since $f = \frac{v_n}{2\pi r_n}$, we have:

$$\begin{aligned} f_1 &= \frac{v_1}{2\pi r_1} \\ &= \frac{Z\alpha c}{2\pi \frac{a_0}{Z}} \\ &= \frac{Z^2 \alpha c}{2\pi a_0} \end{aligned}$$

Substituting back in for a_0 and α :

$$\begin{aligned} f_1 &= \frac{Z^2 \alpha c}{2\pi a_0} \\ &= \frac{Z^2 c \frac{ke^2}{\hbar c}}{2\pi \frac{\hbar^2}{mke^2}} \\ &= Z^2 c \frac{ke^2 mke^2}{2\pi \hbar^3 c} \\ &= Z^2 \frac{mk^2 e^4}{2\pi \hbar^3} \end{aligned}$$

Since $f_1 = f_{rev}$ in this case, we can substitute back in to $i = e f_{rev} = e Z^2 \frac{mk^2 e^4}{2\pi \hbar^3} = Z^2 \frac{mk^2 e^5}{2\pi \hbar^3} = 0.00105 \cdot Z^2 \text{ A}$. We are probably working with hydrogen, since we are looking at the Bohr model, so $Z = 1$, giving us $i = 0.00105 \text{ A}$.

Part b

The magnetic moment of a current loop is iA , where A is the area of the loop. Find the magnetic moment of the electron in the first Bohr orbits in units $\text{A} \cdot \text{m}^2$. This magnetic moment is called a Bohr magneton.

We have $iA = Z^2 \frac{mk^2 e^5}{2\pi \hbar^3} A$ for the magnetic moment. A in this case is just the area of the electron's orbit. This is given by $A = \pi r_1^2$, where $r_1 = \frac{a_0}{Z} = \frac{\hbar^2}{Zmke^2}$. So, we have:

$$\begin{aligned} iA &= Z^2 \left(\frac{mk^2 e^5}{2\pi \hbar^3} \right) \pi \left(\frac{\hbar^2}{Zmke^2} \right)^2 \\ &= Z^2 \left(\frac{mk^2 e^5}{2\pi \hbar^3} \right) \pi \left(\frac{\hbar^4}{Z^2 m^2 k^2 e^4} \right) \\ &= \left(\frac{e}{2} \right) \left(\frac{\hbar}{m} \right) \\ &= \frac{e\hbar}{2m} = 9.27400793 \times 10^{-24} \text{ m}^2 \text{ A} \end{aligned}$$

Problem 4.56

If electric charge did not exist and electrons were bound to protons by the gravitational force to form hydrogen, derive the corresponding expressions for a_0 and E_n and compute the energy and frequency of the H_α line and limit the Balmer series. Compare these with the corresponding quantities for "real" hydrogen.

We start with $F = \frac{GMm}{r^2} = \frac{mv^2}{r}$ and $L = mvr = n\hbar$. We can then solve for v , giving us $v = \sqrt{\frac{GM}{r}}$. Combining this with the equation for L , we get

$$\begin{aligned} n\hbar &= mr \sqrt{\frac{GM}{r}} \\ n\hbar &= m \sqrt{GM r} \\ n^2 \hbar^2 &= m^2 GM r \\ \frac{n^2 \hbar^2}{m^2 GM} &= r \end{aligned}$$

In this case, I would say that $a_0 = \frac{\hbar^2}{Mm^2 G}$. This gives $a_0 = 1.20075776 \times 10^{29} \text{ m}$. So, we have:

$$r_n = n^2 a_0$$

We also have the the total energy of the system would be $E = \frac{1}{2}mv^2 + \left(-\frac{GMm}{r}\right)$. In addition, we can find, from the equation for F , that $\frac{1}{2}mv^2 = \frac{GMm}{2r}$. So, we get the total energy is $E = \frac{GMm}{2r} - \frac{GMm}{r} = -\frac{GMm}{2r}$. Substituting in for r_n we get:

$$\begin{aligned} E_n &= -\left(\frac{GMm}{2}\right)\left(\frac{m^2GM}{n^2\hbar^2}\right) \\ &= -\frac{G^2M^2m^3}{2n^2\hbar^2} \end{aligned}$$

I'm going to say that $E_0 = \frac{G^2M^2m^3}{2\hbar^2}$. This gives us $E_0 = 2.64248168 \times 10^{-78} \text{eV}$. So, we have

$$E_n = -\frac{E_0}{n^2}$$

To find the energy and frequency of H_α , that is the transition from $n = 3$ to $n = 2$, we have:

$$\begin{aligned} E_{H_\alpha} &= E_3 - E_2 \\ &= -\frac{E_0}{3^2} + \frac{E_0}{2^2} \\ &= E_0\left(\frac{1}{4} - \frac{1}{9}\right) \\ &= E_0\left(\frac{5}{36}\right) \\ &= 3.67 \times 10^{-79} \text{eV} \end{aligned}$$

The frequency, then, is $hf = E_3 - E_2 = E_{H_\alpha} = 3.67 \times 10^{-79} \text{eV}$. So, $f = 3.67 \times 10^{-79} \text{eV} \cdot h = 8.87 \times 10^{-65} \text{s}^{-1}$. The limit to this fictitious Balmer series, then, is:

$$\begin{aligned} E_{limit} &= E_\infty - E_2 \\ &= E_0M^2\left(\frac{1}{2^2} - \lim_{n \rightarrow \infty} \frac{1}{n^2}\right) \\ &= \frac{1}{4}E_0M^2 \\ &= \frac{1}{4}\left(9.45 \times 10^{-25} \frac{\text{eV}}{\text{kg}^2}\right)(1.67 \times 10^{-27} \text{kg})^2 \\ &= 6.606 \times 10^{-79} \text{eV} \end{aligned}$$

For real hydrogen, $E_{H_\alpha} = \left(-\frac{13.6 \text{eV}}{3^2}\right) + \left(\frac{13.6 \text{eV}}{2^2}\right) = 4.911 \text{eV}$, $f = 4.911 \text{eV} \cdot h = 1.19 \times 10^{15} \text{s}^{-1}$, and the $E_{limit} = \frac{13.6 \text{eV}}{4} = 3.4 \text{eV}$. The energies and frequencies for this fictitious hydrogen are much, much smaller than for the real thing.