Problem Set #4 Solutions

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#1. a) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$e^{ix} = 1 + ix - \frac{x^2}{2!} + -i\frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)$

$= \cos x + i \sin x$.

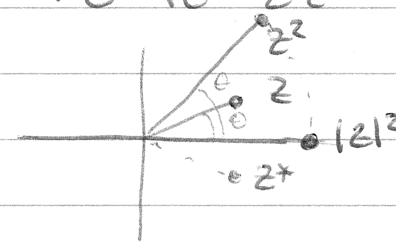
b) $x = r\cos\theta \qquad z = x + iy = r(\cos\theta + i\sin\theta)$

$y = r\sin\theta \qquad = re^{i\theta}$

c) $|z|^2 \equiv z^*z = (x+iy)(x-iy) = x^2 - (iy)^2 = x^2 + y^2 = zz^*$

$= re^{i\theta}\cdot re^{-i\theta} = r^2 e^0 = re^{-i\theta} re^{i\theta} = z^* z$

thus $x^2 + y^2 = r^2$



d) $z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy.$

$= (re^{i\theta})^2 = r^2 e^{i2\theta}$

not real unless z is pure imaginary

e) same as (c) with $r=1$

$1 = e^{i\theta}\cdot e^{-i\theta} = (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = \cos^2\theta + \sin^2\theta$

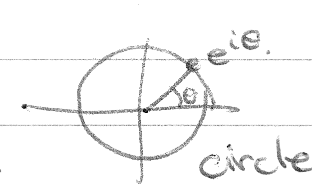
$= e^{i(\theta-\theta)} = e^0 = 1$

thus $e^{i\theta} = x + iy$ where $x^2 + y^2 = 1$, a circle of radius 1

f) $e^{i\theta} = \cos\theta + i\sin\theta \qquad \Rightarrow \qquad \cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$

$e^{-i\theta} = \cos\theta - i\sin\theta \qquad\qquad \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$

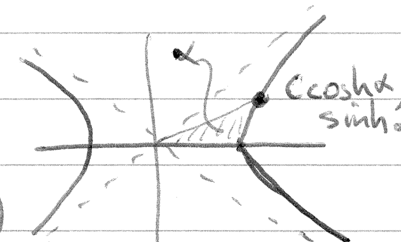
$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})}$



g) $e^{\alpha} = \cosh\alpha + \sinh\alpha$

$e^{-\alpha} = \cosh\alpha - \sinh\alpha$

or $e^{\pm\alpha} = \cosh\alpha \pm \sinh\alpha$



h) $1 = e^0 = e^{\alpha}\cdot e^{-\alpha} = (\cosh\alpha + \sinh\alpha)(\cosh\alpha - \sinh\alpha)$

$1 = \cosh^2\alpha - \sinh^2\alpha. \qquad 1 = x^2 - y^2$ (hyperbola).

i) $e^{i(\alpha\pm\beta)} = e^{i\alpha}\cdot e^{\pm i\beta} = (\cos\alpha + i\sin\alpha)(\cos\beta \pm i\sin\beta)$

Re: $\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

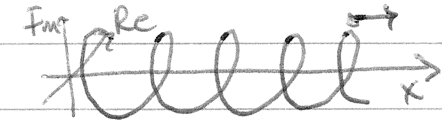
Im: $\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

j) $\frac{d}{d\theta} e^{i\theta} = i e^{i\theta}$

$$\frac{d\cos\theta}{d\theta} + i\frac{d\sin\theta}{d\theta} = i(\cos\theta + i\sin\theta)$$

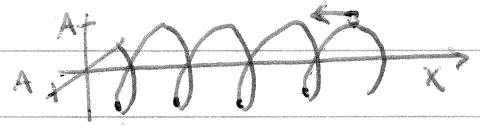
Re: $\frac{d}{d\theta}\cos\theta = -\sin\theta$ Im: $\frac{d}{d\theta}\sin\theta = \cos\theta$.

#2. a) $Ae^{ikx-i\omega t}$



travelling in +x

20 $Ae^{-ikx-i\omega t}$

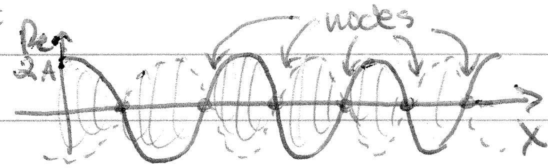


travelling in -x direction

$Ae^{ikx-i\omega t} + Ae^{-ikx-i\omega t}$

$= A(e^{ikx}+e^{-ikx})e^{-i\omega t}$

$= \underbrace{2A\cos(kx)}_{\text{standing wave}}\underbrace{e^{-i\omega t}}_{\text{rotation in complex plane in time}}$



$G e^{-i\omega t}$

"skipping rope wave"

$x = \frac{2\pi}{k}$

It is a standing wave because the nodes always stay in the same place

b) $e^{ik_1x-i\omega_1t} + e^{ik_2x-i\omega_2t}$ let $\theta_1 = k_1x - \omega_1 t$

$= e^{i\theta_1} + e^{i\theta_2}$ $\theta_2 = k_2x - \omega_2 t$

$= e^{i(\bar\theta - \Delta\theta)} + e^{i(\bar\theta+\Delta\theta)}$ let $\bar\theta = \frac{1}{2}(\theta_1+\theta_2)$

$= \cancel{e^{i\bar\theta}}(e^{-i\Delta\theta} + e^{i\Delta\theta})e^{i\bar\theta}$ $\Delta\theta = \frac{1}{2}(\theta_2 - \theta_1)$

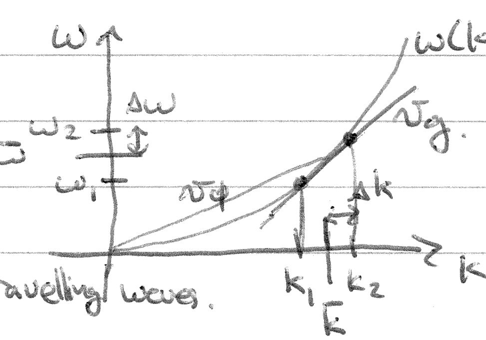
$= 2\cos\Delta\theta \cdot e^{i\bar\theta}$ so $\bar{k} = \frac{1}{2}(k_1+k_2)$

$= \underbrace{2\cos(\Delta k x - \Delta\omega t)}_{\text{group}}\underbrace{e^{i(\bar{k}x - \bar\omega t)}}_{\text{phase}}$ $\Delta k = \frac{1}{2}(k_2 - k_1)$

and $\bar\omega = \frac{1}{2}(\omega_1+\omega_2)$

$v_g = \frac{\Delta\omega}{\Delta k}$ $v_\phi = \frac{\bar\omega}{\bar k}$ $\Delta\omega = \frac{1}{2}(\omega_2 - \omega_1)$

you hear beat freq: $f = \frac{\Delta\omega}{2\pi}$



c) if $\omega_1 = \omega_2$ $k_1 = -k_2$

then $\Delta\omega = 0$ (beat freq.)

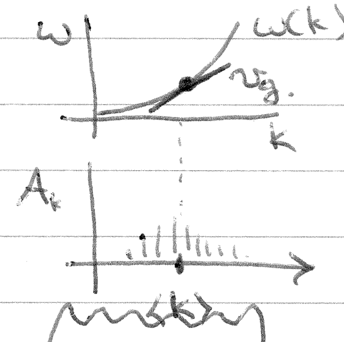
$\Delta k = \frac{1}{2}(k+k) = k$ (standing wave).

has same wavelength as travelling waves.

d) if two waves are very close in wavelength $k_1 \approx k_2$ then $v_g = \lim_{\Delta k \to 0} \frac{\Delta \omega}{\Delta k} \equiv \frac{d\omega}{dk}$ is the derivative.

thus $v_g = \frac{d\omega}{dk}\Big|_{\langle k \rangle}$

for general wave packets.



#3. a) square wave.

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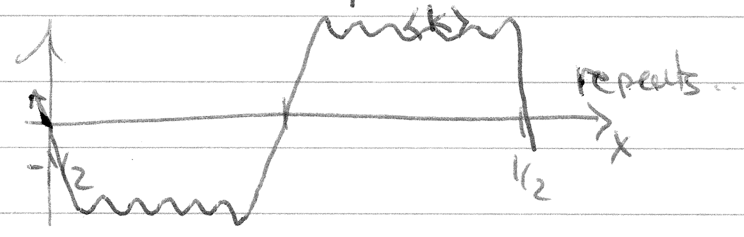

b) $A_1 = 0.5$ $A_3 = 0.78$ $A_5 = 0.41$ $A_7 = 0.17$.

c) - - -

d) Amplitude (k-space) : k_1 = fundamental frequency.
$k_0 = \langle k \rangle$ = average frequency.
$\Delta k = \sigma_k = \sqrt{\langle k^2 \rangle - \langle k \rangle^2}$ = frequency spread.

Sum (x-space) : $\lambda_1 = 2\pi / k_1$ = period of repetition
k_0 = average frequency of phase (carrier wave)
$\sigma_x = \frac{1}{\sigma_k}$ = group (packet) width.

e) as $k_1 \to 0$, the packets spread out to infinity in the limit $k_1 \to 0$ there is a single wave packet. (Fourier integral, continuous $A(k)$), not discrete A_i

f) as $k_0 \to 0$ the carrier wave disappears (long wavelength).
$k_0 \to \infty$ the carrier wave gets higher frequency.

g) as $\sigma_k \to 0$ the packet spreads out $\sigma_x \to \infty$
$\sigma_k \to \infty$ the frequency range spreads out, but the packet gets narrower $\sigma_x \to 0$. (Δx)

h) $\Delta p = \hbar \Delta k$ top plot represents possible momenta.
$\Delta p \cdot \Delta x \geq \hbar/2$ bottom plot represents possible x.

5 #16. a) $\Delta t = \frac{1s}{100\,000} = 10^{-5}s$

b) $\Delta t\, \Delta\omega \geq \frac{1}{2}$ $\Delta\omega \geq \frac{1}{2\Delta t}$ $\Delta f \geq \frac{1}{4\pi \Delta t} = 8\,kHz.$

#25 $\Psi(x,0) = A e^{-\frac{x^2}{4\sigma^2}}$ $\int_{-\infty}^{\infty} |\Psi(x,0)|^2 dx = 1$

$1 = \int_{-\infty}^{\infty} A^2 e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} = A^2 \cdot \sqrt{2\pi\sigma^2}$ $A = (2\pi\sigma^2)^{-1/4}$

10 a) $P = \frac{|\Psi(0,0)|^2 dx}{\sqrt{2\pi}\,\sigma} = \frac{dx}{\sqrt{2\pi}\,\sigma}$

b) $P(\sigma < x < \sigma + dx) = dx \cdot \left(e^{-\frac{\sigma^2}{4\sigma^2}}\right)^2 / \sqrt{2\pi}\,\sigma = \frac{dx}{\sqrt{2\pi e}\,\sigma^2}$

c) $P(2\sigma < x < 2\sigma + dx) = \frac{dx}{\sqrt{2\pi e}} = \frac{dx}{\sqrt{2\pi}\, e^2 \cdot \sigma}$

d) most probable location is at $0 < x < dx$, near center.

5 #27. $\Delta E \geq \frac{\hbar}{2\Delta t} \cdot \frac{c}{c} \frac{197\,eV\cdot nm}{2 \cdot 10^{-7}s \cdot 3\times 10^8 m/s} = \frac{197}{60} neV = 3.28\,neV$

#37. $\Delta p_s\, \Delta s \geq \hbar/2 \Rightarrow r \cdot \Delta p_s \frac{\Delta s}{r} \geq \hbar/2$

10 $\Rightarrow \Delta L_\phi \cdot \phi \geq \hbar/2.$

for $n=1$, $L_\phi = 1 \cdot \hbar$ so $\Delta\phi \geq \frac{1}{2}$ rad $= 29°$.

we will see that $\Delta\phi = 100\%$ for an S-wave.

#41 a) $v_\phi = \frac{\omega}{k} = \frac{E}{p} = \frac{\sqrt{p^2c^2 + m^2c^4}}{p} = \sqrt{c^2\left(1 + \left(\frac{mc}{p}\right)^2\right)} \geq c$ since $\left(\frac{mc}{p}\right)^2 > 0$

10 b) $v_g = \frac{d\omega}{dk} = \frac{dE}{dp}$ $E^2 = p^2c^2 + m^2c^4$

$2E\,dE = 2p\,dp\,c^2$

$v_g = \frac{dE}{dp} = \frac{pc^2}{E} = \frac{\gamma m v c^2}{\gamma m c^2} = v.$

note: $p = \gamma m v$ $p^2c^2 + m^2c^4 = \gamma^2 m^2 v^2 c^2 + m^2 c^4$

$E = \gamma m c^2$ $= m^2c^4\left(\gamma^2 \frac{v^2}{c^2} + 1\right)$

$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$ $= \gamma^2 m^2 c^4 = E^2$