University of Kentucky, Physics 404G
Homework #5, Rev. D, due Monday, 2018-10-08

1. It has been claimed that World War II was won by the Magnetron, a vacuum device which converts electrical energy into microwave radiation. It enabled short-wavelength tactical radar, which used a frequency too high for electrical circuits of the time. Radiation is emitted from several cylindrical microwave cavities tangent to a central cylindrical bore of a solid copper waveguide under vacuum. A thermionic cathode rod of radius a in the center emits electrons, which accelerate from the negative high voltage cathode to the grounded outside walls of the cavity at radius b. A longitudinal magnetic field $\boldsymbol{B}$ sweeps the electrons in a spiral path around the cylinder, exciting resonant currents on the walls of the microwave cavities they pass by. The electric and magnetic fields are perpendicular in this *crossed-field* device.

a) We already calculated the equation of motion for an electron in a magnetic field, $\dot{\eta} = -i\omega\eta$, using complex coordinates $\xi = x + iy$ and $\eta = \dot{\xi} = v_x + iv_y$, where $\omega = qB/m$ is the cyclotron frequency. Add the force $\boldsymbol{F} = q\boldsymbol{E}$ due to the electric field $\boldsymbol{E} = \hat{\boldsymbol{\rho}}V/\rho\ln(b/a)$, where $a = 0.5$ cm and $b = 2.5$ cm are the inner and outer radii of the cylindrical cavity, respectively, with negative voltage V applied to the *cathode* (central electron-emitting elecrode), and the *anode* at ground (0 V). *Hint: the final equations should be of the form* $\dot{\xi} = \eta$ *and* $\dot{\eta} = -i\omega\eta + \lambda\xi/|\xi|^2$, *for some constant* λ *to be determined.*

b) Integrate these equations in Matlab, for an electron emitted from the cathode essentially at rest. Before you start, what will the motion look like qualitatively? Determine the *Hull cut-off* magnetic field B_c, such that an electron barely grazes the surface of the anode and returns to the cathode, using $V = -4$ kV. [bonus:] Determine the function $B_c(V)$.

c) The electrons from part a) excite eight cylindrical resonant cavities of radius $r = 1$ cm distributed around the central cavity, each one separated from the anode (at $\rho = b$) by a $w = 4$ mm long, $t = 5$ mm wide channel cut in the solid cylindrical copper block of thickness d. Calculate the resonant frequency using the inductance $L = N^2\mu_0\pi r^2/d$ and capacitance $C = \epsilon_0 wd/t$. Optimize both the voltage V_0 and magnetic field B such that the electrons just graze past the resonators at the resonant frequency. *Hint: use conservation of energy to determine the required voltage and then return the magnetic field. Warning: Don't try this at home without also including the space-charge effect and its associated "pinwheel"!*

2. Phase space of a Harmonic Oscillator—All of our computer simulations have converted second order ODEs involving $\ddot{x}$ into first order equations in $u = (x, v)$. This is formalized in a more symmetric way by Hamilton's equations of the canonical conjugate variables $\boldsymbol{a} = (x, p)$, as discussed in class.

a) The Lagrangian for the harmonic oscillator is $\mathcal{L} = T - V = \frac{1}{2}mv^2 - \frac{1}{2}kx^2$, where $v = \dot{x}$. Calculate the Canonical momentum $p = \partial\mathcal{L}/\partial v$, and solve Lagrange's equation of motion on x.

b) Calculate $\mathcal{H} = p\dot{q} - \mathcal{L}$ and solve Hamilton's equations by determining the time derivative of the complex coordinate $a = \frac{1}{\sqrt{2}}(x\sqrt{k} + ip/\sqrt{m})$, the same pattern used for cyclotron motion.

c) Plot the trajectory a in the complex plane. Show that $\mathcal{H} = a^*a$. Note that the polar representation $|a|e^{i\phi}$ is called *action-angle* coordinates, an alternative to (x, p) with trivial equations of motion. In quantum mechanics a is made dimensionless by factoring out $\sqrt{\hbar\omega}$ so that $\mathcal{H} = \hbar\omega(a^\dagger a + \frac{1}{2})$.

3. There is a direct mechanical-electrical impedance analogy between electrical circuits and mechanical oscillators, matching each quantity in the drag force $F = vb$ with a corresponding quantity in Ohm's law $V = IR$, which allows direct computation of one system from the solution of the other.

a) What are the analogs of capacitance $Q = CV$ and inductance $V = L\dot{I}$? Write the Lagrangian for an LC circuit, and calculate the canonical momentum of Q. Find the resonant frequency by analogy with the solution for a mass m on a spring k.