University of Kentucky, Physics 404G
Homework #6, Rev. B, due Tuesday, 2018-10-16

1. Pendulum A mass m swings at the end of a massless rod ℓ rotating about a fixed point at the top.

a) Calculate the Lagrangian $\mathcal{L}(\dot{\theta}, \theta, t)$ and resulting equation of motion for the undamped case. Express it in terms of the natural frequency $\omega_0 = \sqrt{g/\ell}$ at small angles. [bonus:] Calculate the exact trajectory, starting from rest at angle θ_0, in terms of the Jacobi elliptic function $sn(u;m)$ and calculate the frequency $\omega(\theta_0)$.

b) Calculate the canonical momentum p_θ and Hamilton's equations for the undamped case. Calculate the maximum velocity of v_1 of the pendulum, starting from rest at the top, using $m = 1$ kg and $\ell = 1$ m. Plot trajectories in phase space starting from equilibrium $\theta = 0$ with initial velocity $v_k = kv_1$ for $k = 0, \pm\frac{1}{5}, \pm\frac{2}{5}, \frac{3}{5}, \ldots 2$. You can obtain a more complete flow of phase space by starting from more equilibrium points: $\theta_0 = 0, \pm 2\pi, \pm 4\pi$.

c) Recalculate the trajectories, adding a frictional force $F = -bv$, starting with $b = 1$ kg/s, normalized to $2\beta = b/m$. Determine the critical damping coefficient b_c, above which oscillations entirely disappear. Repeat the phase space plots for $b = b_c/2, b_c$, and $2b_c$.

d) Recalculate the trajectories, adding a driving force $F = -F_0 \cos(\omega t)$ with $F_0 = \gamma mg$, and $\gamma = 1$ for this problem. Repeat the trajectories $\theta(t)$ and phase space diagrams $[p_\theta, \theta](t)$ for the same values of b as part c).