University of Kentucky, Physics 404G
Homework #7, Rev. B, due Friday, 2018-10-26

0. [bonus:] In H03, we determined the Lagrangian of a double-pendulum, and numerically determined the two modes of oscillation. Use the small-angle approximation to put the Lagrangian in the form $\mathcal{L} = \frac{1}{2}\dot{\boldsymbol{x}}^T\boldsymbol{M}\dot{\boldsymbol{x}} - \frac{1}{2}\boldsymbol{x}^T\boldsymbol{K}\boldsymbol{x}$, where $\boldsymbol{x} = \binom{\alpha}{\beta}$ are the pendulum angles, and $\boldsymbol{M}$ and $\boldsymbol{K}$ are the 2×2 mass and stiffness matrices, respectively. Use Lagrange's equation to obtain $\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = 0$. Solve this equation, using $\boldsymbol{x} = \boldsymbol{a}e^{i\omega t}$ and finding the eigenvalues of $\boldsymbol{K}$.

1. Buttery Slinky—Suspend three identical 1/4 lb cubes of butter from strings of equal height ℓ and horizontal separation d. Couple them with a Slinky draped across the cubes of butter with the same number coils between each cube.

a) Measure the three resonant frequencies of oscillation $\omega_{1,2,3}$ by vibrating one end by hand at different frequencies until the motion becomes periodic. For each frequency, sketch the mode of oscillation, or relative velocity of each cube of butter. Note that the proportionality between the velocities remains constant; they all get larger and smaller together.

b) Measure the mass of the Slinky and the length of a segment of the Slinky hanging under its own weight to determine the spring constant. Calculate the Lagrangian for this system, and use the small-angle approximation to put it in the form $\mathcal{L} = \frac{1}{2}\dot{\boldsymbol{x}}^T\boldsymbol{M}\dot{\boldsymbol{x}} - \frac{1}{2}\boldsymbol{x}^T\boldsymbol{K}\boldsymbol{x}$. Use Lagrange's equation to obtain $\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = 0$, and solve this equation by substituting $\boldsymbol{x} = \boldsymbol{a}e^{i\omega t}$ and finding the eigenvalues (frequencies) and eigenvectors (normal modes) of $\boldsymbol{K}$. [bonus:] Do the same for the systems of one and two masses, respectively.

c) Use the distributed mass and stiffness of the Slinky to show that the equations of motion become the partial differential equation (PDE) $\left(\mu\partial_t^2 - \kappa\partial_x^2\right) f(x,t) = 0$, where $dm = \mu\, dx$ is the mass of a small segment and $\kappa = kd$ is the relative spring constant, i.e. $F = \kappa\, df/dx$. Show that solutions have the form $f_k(x,t) = e^{i(kx-\omega t)}$, by substituting $f_k(x,t)$ into the *wave equation*, and determine the *dispersion relation* between k and ω. The general solution is a linear combination of waves of all wavelength $f(x,t) = \int dk\, C(k)\, f_k(x,t)$, which is the *Fourier transform* of the *frequency spectrum* $C(k) = A(k)e^{i\phi(k)}$. Compare with the measured velocity of the longitudinal waves on the Slinky.

2. Waves on a string of linear mass density μ stretched horizontally with tension T.

a) Derive the wave equation for a mass on a string. *Hint: apply Newton's law, ignoring gravity, to a segment of string of infinitesimal length dx and mass dm.*

b) Substitute the wave function $f(x,t) = Ae^{i(kx-\omega t)}$ into the wave equation to derive the dispersion relation $\omega(k)$. Use this to calculate the phase velocity of waves on the string.

c) A string of length L is fixed at one end. The other end is tied to a massless ring, sliding freely up and down a rod. Use boundary conditions to find the spectrum of allowed frequencies.

d) In analogy with AC electrical impedance $Z = V/I = R + i\omega L + 1/i\omega C$, the mechanical impedance of an oscillating system (mass M, spring constant K, drag coefficient B) is $Z = F/v = B + i\omega M + K/i\omega$. Calculate the characteristic impedance of the string, $Z = F/v = Tf'/\dot{f}$ (the ratio of vertical force to velocity) in terms of T and μ. Show the power transferred along the wave is $P = Z\dot{f}^2$. [bonus:] Show that the energy density is $U = Tf'^2 = \mu\dot{f}^2$, and therefore $P = Uv$, meaning that energy is transferred along the wave at velocity v.

e) Two strings of density $\mu_{1,2}$ and tension $T_{1,2}$ are joined by a ring on a rod to support the difference in tension. Justify the boundary conditions $\Delta f = 0$ and $\Delta(Tf') = \Delta(\pm Z\dot{f}) = Z_0\dot{f}$, where $\pm$ is positive[negative] for a forward[backward] traveling wave $e^{\pm ikx}$, respectively, and Z_0 is the impedance of the ring. *Hint: apply Newton's law to vertical forces on the ring to obtain the second condition.*

f) An incident wave $Ae^{i(k_1x-\omega t)}$ from the left ($x<0$) is partially reflected at the ring at $x=0$ due to the change in impedance of the two strings. The reflected wave $Be^{i(-k_1x-\omega t)}$ is superimposed on the incident wave for $x<0$, while the forward transmitted wave is $Fe^{i(k_2x-\omega t)}$ for $x>0$. Apply the two boundary conditions to obtain the reflected amplitude B and transmitted amplitude F as a function of the fixed incident amplitude A. Calculate the coefficients of reflected power $R = Z_1B^2/Z_1A^2$ and transmitted power $T = Z_2F^2/Z_1A^2$ in terms of $Z_{1,2}$. Show they add up to 100%.

g) [bonus:] Repeat part f) for the two strings joined by a ring of impedance Z_0 (mass M, spring constant K, and damping B).