University of Kentucky, Physics 404G
Homework #11, Rev. A, [bonus]

1. Physical interpretation of the **vector potential $\boldsymbol{A}$**. The quantity $q\boldsymbol{A}$ has physical significance as potential momentum, just as qV is potential energy. The purpose of this problem is to explore the implications of this connection. For more background information see Am. J. Phys. **64**, 1368 (1996).

a) Show that $\partial\rho/\partial t + \nabla\cdot\boldsymbol{J} = 0$ (conservation of charge). In 4-vector notation, this is written $\partial_\mu J^\mu = 0$, where $\partial = (\partial_t, \nabla)$, and $J = (\rho, \boldsymbol{J})$.

b) For a continuous distribution $\rho(\boldsymbol{r}, t)$ of charges in a velocity field $\boldsymbol{v}(\boldsymbol{r}, t)$ so that $\boldsymbol{J} = \rho\boldsymbol{v}$, show that the *convective derivative* of ρ (the time derivative of $\rho(\boldsymbol{r}(t), t)$ along the path $\boldsymbol{r}(t)$ with velocity $\boldsymbol{v} = d\boldsymbol{r}/dt$) is $d\rho/dt = \partial\rho/\partial t + \boldsymbol{v}\cdot\nabla\rho$, where $\partial\rho/\partial t$ is the time derivative of ρ at a fixed position $\boldsymbol{r}$ and $\nabla\rho$ is the gradient of ρ at a fixed time. Use part a) to show that $d\rho/dt + \rho\nabla\cdot\boldsymbol{v} = 0$. What is the physical interpretation of $\nabla\cdot\boldsymbol{v}$ for a velocity field?

c) Show that using the velocity-dependent potential $U = q(V - \boldsymbol{v}\cdot\boldsymbol{A})$ in Lagrange's equation produces the Lorentz force:

$$\boldsymbol{F} \equiv \frac{d}{dt}\frac{\partial U}{\partial \boldsymbol{v}} - \frac{\partial U}{\partial \boldsymbol{r}} = q(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}), \tag{1}$$

where $\partial/\partial\boldsymbol{r} = \nabla = \hat{\boldsymbol{x}}\partial_x + \hat{\boldsymbol{y}}\partial_y + \hat{\boldsymbol{z}}\partial_z$ and $\partial/\partial\boldsymbol{v} = \nabla_{\boldsymbol{v}} = \hat{\boldsymbol{x}}\partial_{v_x} + \hat{\boldsymbol{y}}\partial_{v_y} + \hat{\boldsymbol{z}}\partial_{v_z}$ are gradients with respect to the independent variables $\boldsymbol{r}$ and $\boldsymbol{v}$, respectively. Use $\boldsymbol{E} \equiv -\nabla V - \partial\boldsymbol{A}/\partial t$, which includes time dependence. This generalizes the standard formula $\boldsymbol{F} = -\partial U/\partial\boldsymbol{r} = -\nabla U$. Thus the electromagnetic Lagrangian is $L = T - U = \frac{1}{2}mv^2 - q(V - \boldsymbol{v}\cdot\boldsymbol{A})$.

d) Calculate the *canonical momentum* $\boldsymbol{\pi} \equiv \frac{\partial L}{\partial \boldsymbol{v}}$. This generalized momentum includes contributions from both the particle and the magnetic field.

e) Calculate the *Hamiltonian* $H \equiv \boldsymbol{\pi}\cdot\boldsymbol{v} - L$. Express H in terms of $\boldsymbol{\pi}$ and $\boldsymbol{r}$, not $\boldsymbol{v}$. This is the total energy in the system. Thus the Hamiltonian and canonical momentum $(H, \boldsymbol{\pi})$ are the generalized energy and momentum of a system. Each includes a kinetic and potential term.

f) Show that $\frac{dH}{dt} = \frac{d}{dt}(T + qV) = \frac{\partial}{\partial t}q(V - \boldsymbol{v}\cdot\boldsymbol{A}) = \frac{\partial U}{\partial t}$, where $\frac{\partial}{\partial t}$ acts on the potentials, but not $\boldsymbol{r}$ or $\boldsymbol{v}$. Thus the total energy H is conserved if the generalized potential is time-independent.

g) Show that $\frac{d\boldsymbol{\pi}}{dt} = \frac{d}{dt}(\boldsymbol{p} + q\boldsymbol{A}) = -\frac{\partial}{\partial \boldsymbol{r}}q(V - \boldsymbol{v}\cdot\boldsymbol{A}) = -\frac{\partial U}{\partial \boldsymbol{r}}$, where again $\frac{\partial}{\partial \boldsymbol{r}}$ acts on the potentials, but not $\boldsymbol{v}$. Thus $\boldsymbol{\pi}$ is conserved if the generalized potential is position-independent.