University of Kentucky, Physics 404G
Homework #2, Rev. A, due Thursday, 2019-09-19

1. Projections: relational versus parametric geometry.

a) Show graphically that the following equations define a set of points $\{\boldsymbol{x}\}$ on a **line** or **plane**,

	relational	parametric
line	$\{\boldsymbol{x} \mid \boldsymbol{a} \times \boldsymbol{x} = \boldsymbol{d}\}$	$\{\boldsymbol{x} = \boldsymbol{x}_1 + \boldsymbol{a}\alpha \mid \alpha \in \mathbb{R}\}$
plane	$\{\boldsymbol{x} \mid \boldsymbol{A} \cdot \boldsymbol{x} = D\}$	$\{\boldsymbol{x} = \boldsymbol{x}_2 + \boldsymbol{b}\beta + \boldsymbol{c}\gamma \mid \beta, \gamma \in \mathbb{R}\}$

where $\boldsymbol{a}, \boldsymbol{d}, \boldsymbol{A}, D$ are constants which define the geometry, $\boldsymbol{x}_{1,2}$ are fixed points on the line and plane respectively, and α, β, γ are parameters that vary along the line/plane (they uniquely parametrize points in the line/plane). [bonus:] Show the 5th relation $\{\boldsymbol{x} = \boldsymbol{x}_2 + \boldsymbol{A} \times \boldsymbol{\delta} \mid \boldsymbol{\delta} \in \mathbb{R}^3\}$.

b) What constraint between $\boldsymbol{a}$ and $\boldsymbol{d}$ is implicit in the formula $\boldsymbol{a} \times \boldsymbol{x} = \boldsymbol{d}$? Likewise, what relations between $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{A}$ must be satisfied if both equations describe the same plane? For the line and the plane, substitute $\boldsymbol{x}$ from the parametric form into the relational form to show that they are consistent. What are $\boldsymbol{d}$ and D in terms of $\boldsymbol{a}, \boldsymbol{x}_1$ and $\boldsymbol{A}, \boldsymbol{x}_2$, respectively?

c) We define $\tilde{\boldsymbol{a}} \equiv \boldsymbol{A}/(\boldsymbol{a} \cdot \boldsymbol{A})$. It is parallel to $\boldsymbol{A}$ but *normalized* in the sense that $\boldsymbol{a} \cdot \tilde{\boldsymbol{a}} = 1$. Using the BAC-CAB rule, show that $\boldsymbol{x} = \boldsymbol{a}(\tilde{\boldsymbol{a}} \cdot \boldsymbol{x}) - \tilde{\boldsymbol{a}} \times (\boldsymbol{a} \times \boldsymbol{x})$ for any $\boldsymbol{x}$. This is a non-orthogonal projection of $\boldsymbol{x}$ into a vector parallel to the line and a vector parallel to the plane. Illustrate this projection and show which term corresponds to each.

d) [bonus:] Using c), calculate the point $\boldsymbol{x}_0$ at the intersection of the line and plane in terms of $\boldsymbol{a}, \boldsymbol{d}, \boldsymbol{A}, D$. Verify this by showing $\boldsymbol{x}_0$ satisfies the relational equation for both the line and plane.

e) Let $\tilde{\boldsymbol{a}} = \frac{\vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}}}{\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}}}$, $\tilde{\boldsymbol{b}} = \frac{\vec{\boldsymbol{c}} \times \vec{\boldsymbol{a}}}{\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}}}$, and $\tilde{\boldsymbol{c}} = \frac{\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}}{\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}}}$, where the arrows on $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ distinguish $\vec{\boldsymbol{a}}, \vec{\boldsymbol{b}}, \vec{\boldsymbol{c}}$ from their *covectors* $\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}}, \tilde{\boldsymbol{c}}$. This definition of $\tilde{\boldsymbol{a}}$ is consistent with above if $\boldsymbol{A} = \vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}}$. Calculate the nine combinations of $(\vec{\boldsymbol{a}}, \vec{\boldsymbol{b}}, \vec{\boldsymbol{c}})^T \cdot (\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}}, \tilde{\boldsymbol{c}}) = I$, i.e. $\vec{\boldsymbol{a}} \cdot \tilde{\boldsymbol{a}} = 1$, $\vec{\boldsymbol{a}} \cdot \tilde{\boldsymbol{b}} = 0$, ... to show they are *mutually orthogonal*. In this sense, $(\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}}, \tilde{\boldsymbol{c}})$ is the *dual* or *reciprocal* basis of $(\vec{\boldsymbol{a}}, \vec{\boldsymbol{b}}, \vec{\boldsymbol{c}})$.

f) The *contravariant components* of $\boldsymbol{x}$ are defined as the components (α, β, γ) that satisfy the equation $\boldsymbol{x} = \vec{\boldsymbol{a}}\alpha + \vec{\boldsymbol{b}}\beta + \vec{\boldsymbol{c}}\gamma$; i.e., $(\vec{\boldsymbol{a}}, \vec{\boldsymbol{b}}, \vec{\boldsymbol{c}})$ is the contravariant basis. Using $\boldsymbol{x} \cdot \tilde{\boldsymbol{a}}$, etc., calculate the three contravariant components of $\boldsymbol{x}$ in terms of dot products. Likewise, find the *covariant components* $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ of $\boldsymbol{x}$, defined as the components $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ which satisfy the equation $\boldsymbol{x} = \tilde{\boldsymbol{a}}\tilde{\alpha} + \tilde{\boldsymbol{b}}\tilde{\beta} + \tilde{\boldsymbol{c}}\tilde{\gamma}$; i.e. $(\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}}, \tilde{\boldsymbol{c}})$ is the covariant basis.

2. Rotations. In addition to projection, the cross product *generates rotation*. For example, $\boldsymbol{\omega}\times$ transforms to a rotating reference frame and $\boldsymbol{r}\times$ appears in torque and angular momentum. Rotations are ubiquitous in all of physics, particularly mechanics. We will explore parallels between the cross product, the unit imaginary i, and matrix generator M in the context of rotations.

a) The *complex plane* $\{\boldsymbol{w} = (x, y)\}$ is the vector space of *real* and *imaginary* components of complex numbers $w = x + iy \in \mathbb{C}$, where $i^2 = -1$. Identify the two natural *basis vectors* (complex numbers) in this space. In addition to linear structure, show that complex multiplication has the form $w = w_1 w_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2) = x + iy$, matching matrix products of $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$.

b) Show that the dot and cross product of two points $\boldsymbol{w}_1 = (x_1, y_1)$ and $\boldsymbol{w}_2 = (x_2, y_2)$ are given by the real and imaginary parts of the complex product $w_1^* w_2 = \boldsymbol{w}_1 \cdot \boldsymbol{w}_2 + i(\boldsymbol{w}_1 \times \boldsymbol{w}_2)_z$, where

$w^* = x - iy$ is the *complex conjugate* of w. Identify the symmetric and antisymmetric terms of this product. Thus the complex product $|w|^2 = w^* w$ equals the vector square $\boldsymbol{w} \cdot \boldsymbol{w}$.

c) Show graphically that the operator $w \to iw$ rotates the point w 90° CCW about the origin, and that the operator $1 + i\,d\phi : w \mapsto w + iw\,d\phi$ preserves the magnitude of w (assuming $d\phi^2 = 0$), but rotates it CCW by the infinitesimal angle $d\phi$.

d) Obtain a finite rotation from an infinite number of $d\phi$ rotations as follows: formally integrate the equation $dw = iw\,d\phi$ with the initial condition $w|_{\phi=0} = w_0$ to obtain the rotation formula $w(\phi) = R_\phi w_0$, where $R_\phi = e^{i\phi}$. Use this result to justify the identity $e^x = \lim_{n\to\infty}(1 + \frac{x}{n})^n$. Show this is a rotation by separating the Taylor expansion of $e^{i\phi}$ into $x + iy$ to prove Euler's formula, $e^{i\phi} = \cos\phi + i\sin\phi$.

e) Show that complex multiplication by i is equivalent to the vector operator $\hat{\boldsymbol{z}}\times$, and determine the matrix representation M_z of this operator; ie. $M_z \boldsymbol{r} = \hat{\boldsymbol{z}} \times \boldsymbol{r}$. Do the same for M_x and M_y to show that $\boldsymbol{v}\times = \boldsymbol{v} \cdot \boldsymbol{M} = v_x M_x + v_y M_y + v_z M_z$ is the matrix representation of $\boldsymbol{v}\times$ for any vector $\boldsymbol{v}$. *Answer:* $\boldsymbol{M} \sim (M_i)_{jk} = \varepsilon_{ijk}$, the Levi-Civita (cross product), tensor which is completely antisymmetric in indices i, j, k.

f) Restricted to the xy-plane, show that $M_z^2 = -I$, a 2×2 matrix, analogous to $i^2 = -1$, and thus the matrix for a CCW rotation ϕ is $R_\phi = e^{M_z\phi} = I\cos\phi + M_z \sin\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$. Note that the *exponential* of a matrix M_z is defined by its Taylor expansion. The *Rodrigues formula* for the matrix $R_{\boldsymbol{v}}$ of a CCW rotation by an angle $v = |\boldsymbol{v}|$ about the $\hat{\boldsymbol{v}}$-axis is $R_{\boldsymbol{v}} = e^{\boldsymbol{M}\cdot\boldsymbol{v}} = I\cos v + M\cdot\hat{\boldsymbol{v}}\sin v + \hat{\boldsymbol{v}}\hat{\boldsymbol{v}}^T(1 - \cos v)$. The third term projects out the axis of rotation $\hat{\boldsymbol{v}}$.

3. Stretches. In analogy with the *polar decomposition* $w = x + iy = \rho e^{i\phi}$ of complex numbers, any matrix A can be decomposed $A = RS$ into a stretch S and a rotation R, which are the building blocks of all linear operators.

a) Use the exponential form $R_{\boldsymbol{v}} = e^{\boldsymbol{M}\cdot\boldsymbol{v}}$ and the asymmetry of the generator $M^\dagger = -M$ to show that $R^\dagger R = I$ or $R^\dagger = R^{-1}$. In contrast, a stretch is symmetric or *Hermitian* $S^\dagger = S$.

b) A symmetric matrix S is guaranteed to have a complete set of eigenvectors $V = (\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots)$ and corresponding eigenvalues $\lambda_1, \lambda_2, \ldots$, such that $S\boldsymbol{v}_i = \boldsymbol{v}_i\lambda_i$. Show that we can extend this equation to $SV = VW$, where $W = \mathrm{diag}(\lambda_1, \lambda_2, \ldots)$ is *diagonal.*

c) Show that two eigenvectors $\boldsymbol{v}_i, \boldsymbol{v}_j$ of S with distinct eigenvalues $\lambda_i \neq \lambda_j$ must be orthogonal: $\boldsymbol{v}_i \cdot \boldsymbol{v}_j = 0$, and therefore the matrix of eigenvectors is an orthogonal transformation: $V^\dagger V = I$. [bonus]: Interpret the resulting decompositions $S = VWV^\dagger$ and $W = V^\dagger S V$ in terms of rotations.

d) Calculate the eigenvalues and eigenvectors of $M_z = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ Do they look familiar? Show that $M_z = VWV^\dagger = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$ and $e^{M_z\phi} = Ve^{W\phi}V^\dagger = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$. Multiply this out to verify part 2f). Thus real Hermitian matrices have real eigenvalues while antiHermitian matrices have Tr=0 and imaginary eigenvalues. The exponential of a Hermitian matrix is positive definite with real positive eigenvalues, while the exponential of an antiHermitian matrix is unitary with Det=1 and unit modulus eigenvalues. This is the normal matrix analogy which completes the connection between matrices and complex numbers.

e) *Singular Value Decomposition (SVD)* [bonus] Combine polar ($A = RS$) and eigen ($S = VWV^\dagger$) decompositions, using $U = RV$, to show that any $m \times n$ matrix ${}_mA_n$ may be decomposed into $A = UWV^\dagger$, where $U \equiv V^\dagger R$, $U^\dagger U = I_m$, $V^\dagger V = I_n$, and ${}_mW_n$ is diagonal and *positive definite.*