

University of Kentucky, Physics 404G  
Homework #3, Rev. C, due Tuesday, 2019-10-01

1. The **Cartesian**, **cylindrical**, and **spherical** coordinate systems are by far the most common. Do parts a),b),c), and e) for each of these coordinate systems, d) for cylindrical only:

a) Determine the coordinate transformation functions  $q^i(\mathbf{r})$  in terms of each of the other two systems. *Hint: combine the simpler Cartesian-cylindrical and cylindrical-spherical transformations to obtain Cartesian-spherical.*

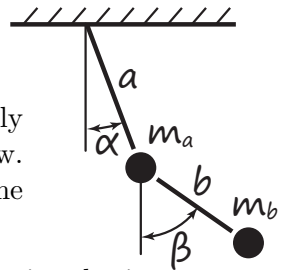
b) Pick a point  $\mathbf{r}_0$  and illustrate the three coordinate isosurfaces  $q^i(\mathbf{r}) = q_0^i$  (constant) that pass through  $\mathbf{r}_0$ , labeling all lengths and angles in your diagram. For each coordinate  $q^i$ , identify the curve  $\mathbf{s}(q^i; q_0^j, q_0^k)$  at the intersection of two surfaces of constant  $q^j = q_0^j$  and  $q^k = q_0^k$ .

c) Calculate  $\mathbf{b}_i = \partial \mathbf{s} / \partial q^i$  using  $d\mathbf{s} = \hat{x}dx + \hat{y}dy + \hat{z}dz$ . Normalize  $\mathbf{b}_i = \hat{\mathbf{e}}_i h_i$  to find the *unit vectors* and *line element*  $d\mathbf{s} = \hat{\mathbf{e}}_i h_i dq^i$ , [bonus: *area element*  $d\mathbf{a} = \frac{1}{2} d\mathbf{s} \times d\mathbf{s} = \hat{\mathbf{e}}_k h_i h_j dq^i dq^j$  for  $i, j, k$  cyclic], and [bonus: *volume element*  $d\tau = \frac{1}{3} d\mathbf{s} \cdot d\mathbf{a} = h_1 h_2 h_3 dq^1 dq^2 dq^3$  (for orthogonal coordinates)]. Note that the scale factors  $h_\theta$  and  $h_\phi$  for angular coordinates are just radius of curvature, according to the arc length formulae  $ds_\theta = r d\theta$  and  $ds_\phi = \rho d\phi$ .

d) Invert cylindrical coordinates to obtain  $(\rho, \phi, z) = f^{-1}(x, y, z)$ . Calculate the covariant basis  $\mathbf{b}^i = \nabla q^i = \hat{\mathbf{e}}_i / h_i$  to verify  $\mathbf{b}_i \cdot \mathbf{b}^j = \delta_i^j$ . [bonus: Same for spherical coordinates].

e) Calculate  $g_{ij} = \mathbf{b}_i \cdot \mathbf{b}_j = \text{diag}(h_1^2, h_2^2, h_3^2)$  and [bonus:  $\Gamma_{ij} = \partial \mathbf{b}_i / \partial q^j$ ].

2. The position of the lower mass  $m_b$  of a **double pendulum** is most naturally parametrized by the non-orthogonal coordinate system  $(\alpha, \beta)$ , illustrated below. The position of the upper mass  $m_a$  is trivially parametrized by  $\alpha$ , but that of the lower mass  $m_b$  depends on both angles  $(\alpha, \beta)$ .



a) Calculate and draw the  $\alpha$  and  $\beta$  coordinate lines, the contravariant and covariant basis vectors  $\mathbf{b}_i$  [bonus: and  $\mathbf{b}^i$ ]. Use the components of  $d\mathbf{s}$  to obtain the velocity  $\mathbf{v} = \dot{\mathbf{r}} = \mathbf{b}_i \dot{q}^i$  of the lower mass.

b) Calculate the metric  $g_{ij} = \mathbf{b}_i \cdot \mathbf{b}_j$  and thus  $ds^2 = g_{ij} dq^i dq^j$  to obtain the kinetic energy  $T_b = \frac{1}{2} m_b v^2 = \frac{1}{2} m_b g_{ij} \dot{q}^i \dot{q}^j$  of the lower mass. Raise the indices of  $\mathbf{b}_i$  with the inverse metric  $g^{ij}$  to obtain the covariant basis  $\mathbf{b}^i$ . [bonus: Compare with a direct calculation of the gradient  $\mathbf{b}^i = \nabla q^i$ ].

c) Calculate the connection vectors  $\Gamma_{ij} = \partial \mathbf{b}_i / \partial q^j$ . Compare the covariant components  $\Gamma_{ijk} = \Gamma_{ij} \cdot \mathbf{b}_k$  of  $\Gamma_{\alpha\alpha}$  with those obtained directly from the metric  $\Gamma_{ijk} = \frac{1}{2} (g_{ik,j} + g_{jk,i} - g_{ij,k})$ . Use  $\Gamma_{ijk}$  to obtain the covariant components of acceleration  $\mathbf{A} = \dot{\mathbf{v}} = \mathbf{b}_k \ddot{q}^k + \Gamma_{ij} \dot{q}^i \dot{q}^j$  of the mass  $m_b$ .

d) Calculate the potential of gravity  $V(\alpha, \beta) = mgh$  for both masses  $m_a$  and  $m_b$  in terms of  $(\alpha, \beta)$  to construct the Lagrangian  $\mathcal{L} = T - V$ . Compare the two equations of motion for  $\alpha$  and  $\beta$ , in the covariant components of  $\mathbf{F} = m\mathbf{A}$  using  $F_i = \partial V / \partial q^i$  and  $A_i$  from above, with the direct calculation using Lagrange's equations.

e) Computer exercise: numerically integrate the four first order equations of motion in the variables  $\mathbf{q} = (\alpha, \beta, \dot{\alpha}, \dot{\beta})$  for  $a = 2$  m and  $b = 1$  m,  $m_a = m_b = 1$  kg, starting from initial conditions  $\mathbf{q}_0 = (.1, .2, 0, 0)$ , in units of [rad] and [rad/s]. Vary the initial value  $\beta_0$  to discover two *modes* of pure periodic motion, and calculate their [eigen]frequencies. Increase  $\alpha_0$  and  $\beta_0$  until the motion becomes *chaotic* (unpredictable). *Hint: start with the code for this [simple pendulum](#).*