University of Kentucky, Physics 404G
Homework #4, Rev. A, due Thursday, 2019-10-10

1. Calculus of variations was foreseen by Pierre de Fermat, who realized that *Snell's law* could be viewed as a *principle of least time* (or *optical path length*) for a ray of light to travel between two points. He essentially invented *calculus* to solve this and other types of minimization problems, although he mislabeled the use of his infinitesimal ϵ as *adequality* instead of more correctly as $\lim_{\epsilon \to 0}$. The term "adequality" for "approximate equality" was borrowed from Diophantus, in whose book he penciled *Fermat's last theorem* in the margin without proof.

Maupertuis, who felt that "Nature is thrifty in all of its actions," was bothered that time and distance were not treated on an equal footing. He developed the principle of **least action**, minimizing instead the integral $\int v ds = \int 2T/m \, dt$, one of the first appearances of *vis viva*. This was further expanded from optics to mechanics by Euler as the *abbreviated action* $S_0 = \int p \, dq$. The modern action $S = \int \mathcal{L} dt$ and associated equations of stationary action $(\frac{d}{dt}\frac{\partial}{\partial \dot{q}} - \frac{\partial}{\partial q})\mathcal{L} = 0$ were developed by Lagrange and Hamilton. This same principle extends to quantum field theory and *Feynman path integrals*. We have already seen that these equations are just Newton's law written in covariant components. This problem explores the interpretation of the expression in parenthesis as a variational derivative $\frac{\delta}{\delta q}$ for all path variations δq.

a) Minimize the time taken for a ray of light to travel from $P_1 = (x_1, y_1)$ to $P_2 = (x_2, y_2)$ through an interface at $y = 0$ between two materials of *index of refraction* n_1 and n_2. Assume for now that the shortest path between two points is a straight line, so that the only variable is x, where the ray crosses the interface at $y = 0$. Derive Snell's law from your answer. [bonus: extend your solution to the case of a ray crossing multiple parallel interfaces, and show it is equivalent to $\nabla f(x_1, x_2, \ldots) = 0$ for crossing points x_i.]

b) Use Fermat's *principle of stationary optical path length* to calculate the image of a mirage by minimizing the functional $I[x(y)] = \int n ds = \int n(y)\sqrt{1 + x'(y)} dy$, or setting its *functional derivative* $\frac{\delta I}{\delta x} \equiv (-\frac{d}{dy}\frac{\partial}{\partial x'} - \frac{\partial}{\partial x}) n(y)\sqrt{1 + x'(y)}$ to zero for the path $y(x)$. [Nearing, p. 520] Show the differential form of Snell's law, that $n(y)\frac{x'(y)}{1+x'(y)^2} = n\frac{dx}{ds} = n\sin(\theta) = C$ is constant along the path or, rearranging, that $x'(y) = C/\sqrt{n(y)^2 - C^2}$. To solve this we need to know the index of refraction $n(y)$. Let us assume it is linear with height, $n(y) = n_0 + \alpha y$. Use the substitution $n(y) = C\cosh(\theta)$ to integrate $x'(y)$. Invert the solution of $x(y)$ to obtain $y = \frac{1}{\alpha}(-1 + \frac{C}{n_0}\cosh(\frac{n_0 \alpha}{C}(x - x_0)))$. Note that a chain fixed at both ends has the same shape for a similar reason: it also hangs under a linear potential, $V(y) = V_0 + mgy$, of gravity in this case.

2. Phase space of a Harmonic Oscillator—All of our computer simulations have converted second order ODEs involving $\ddot{x}$ into first order equations in $u = (x, v)$. This is formalized in a more symmetric way by Hamilton's equations of the canonical conjugate variables $\boldsymbol{a} = (x, p)$, as discussed in class.

a) The Lagrangian for the harmonic oscillator is $\mathcal{L} = T - V = \frac{1}{2}mv^2 - \frac{1}{2}kx^2$, where $v = \dot{x}$. Calculate the Canonical momentum $p = \partial\mathcal{L}/\partial v$, and solve Lagrange's equation of motion on x.

b) Calculate $\mathcal{H} = p\dot{q} - \mathcal{L}$ and solve Hamilton's equations by treating the point (x, p) as a complex number $a = \frac{1}{\sqrt{2}}(x\sqrt{k} + ip/\sqrt{m})$ and taking its time derivative.

c) Plot the trajectory a in the complex plane. Show that $\mathcal{H} = a^* a$. Note that the polar representation $|a|e^{i\phi}$ is called *action-angle* coordinates, an alternative to (x, p) with trivial equations of motion. In quantum mechanics a is made dimensionless by factoring out $\sqrt{\hbar\omega}$ so that $\mathcal{H} = \hbar\omega(a^\dagger a + \frac{1}{2})$.

d) Integrate the abbreviate action $S_0 = \int p\, dx$ around one cycle. Note that a circle is the geometric shape with the maximum area, as demanded by Hamilton's *principle of stationary action.*

3. Damped ballistic motion—In H01, we neglected air resistance of our frog-prince's trajectory. Assume a spherical frog of diameter $D = 5$ cm, and mass $m = 1$ kg. **a)** Calculate the Reynolds number $R = Dv\rho/\eta$ for the initial muzzle velocity $v_0 = 10$ m/s, using the viscosity $\eta = 1.7 \times 10^{-5}$ N s/m^2 and density $\rho = 1.29$ kg/m^3 of air at STP, and decide whether to to use a linear $F_{lin} = 3\pi\eta Dv$ or quadratic $F_{quad} = \frac{1}{4}\rho A v^2$ drag force. Calculate the terminal velocity v_{ter}.

b) Repeat the simulation including air resistance, using the same initial conditions as tuned in H01. Does he still make it? Determine the initial velocity and direction required to safely land the frog-prince, accounting for air resistance. [bonus: Plot and compare the trajectories with and without resistance; including both numerical and analytic solutions on the same graph.]