University of Kentucky, Physics 404G
Homework #5, Rev. A, due Friday, 2019-10-18

1. It has been claimed that World War II was won by the **Magnetron**, a vacuum device which converts electrical energy into microwave radiation. It enabled short-wavelength tactical radar, which used a frequency too high for electrical circuits of the time. Radiation is emitted from several cylindrical microwave cavities tangent to a central cylindrical bore of a solid copper waveguide under vacuum. A thermionic cathode rod of radius a in the center emits electrons, which accelerate from the negative high voltage cathode to the grounded outside walls of the cavity at radius b. A longitudinal magnetic field $\boldsymbol{B}$ sweeps the electrons in a spiral path around the cylinder, exciting resonant currents on the walls of the microwave cavities they pass by. The electric and magnetic fields are perpendicular in this *crossed-field* device.

a) We already calculated the equation of motion for an electron in a magnetic field, $\dot{\eta} = -i\omega\eta$, using complex coordinates $\xi = x + iy$ and $\eta = \dot{\xi} = v_x + iv_y$, where $\omega = qB/m$ is the cyclotron frequency. Add the force $\boldsymbol{F} = q\boldsymbol{E}$ due to the electric field $\boldsymbol{E} = \hat{\boldsymbol{\rho}}V/\rho\ln(b/a)$, where $a = 0.5$ cm and $b = 2.5$ cm are the inner and outer radii of the cylindrical cavity, respectively, with negative voltage V applied to the *cathode* (central electron-emitting elecrode), and the *anode* at ground (0 V). *Hint: the final equations should be of the form* $\dot{\xi} = \eta$ *and* $\dot{\eta} = -i\omega\eta + \lambda\xi/|\xi|^2$*, for some constant* λ *to be determined.*

b) Integrate these equations in Matlab, for an electron emitted from the cathode essentially at rest. Before you start, what will the motion look like qualitatively? Determine the *Hull cut-off* magnetic field B_c, such that an electron barely grazes the surface of the anode and returns to the cathode, using $V = -4$ kV. [bonus: Determine the function $B_c(V)$.]

c) The electrons from part a) excite eight cylindrical resonant cavities of radius $r = 1$ cm distributed around the central cavity, each one separated from the anode (at $\rho = b$) by a $w = 4$ mm long, $t = 5$ mm wide channel cut in the solid cylindrical copper block of thickness d. Calculate the resonant frequency using the inductance $L = N^2\mu_0\pi r^2/d$ and capacitance $C = \epsilon_0 wd/t$. Optimize both the voltage V_0 and magnetic field B such that the electrons just graze past the resonators at the resonant frequency. *Hint: use conservation of energy to determine the required voltage and then determine the magnetic field. Warning: Don't try this at home without also including the space-charge effect and its associated "pinwheel"!*

2. Damped Driven Pendulum A mass m swings at the end of a massless rod ℓ rotating about a fixed point at the top.

a) Calculate the Lagrangian $\mathcal{L}(\dot{\theta}, \theta, t)$ and resulting equation of motion for the undamped, undriven case. Express it in terms of the natural frequency $\omega_0 = \sqrt{g/\ell}$ at small angles. [bonus: Calculate the exact trajectory, starting from rest at angle θ_0, in terms of the Jacobi elliptic function $sn(u;m)$ and calculate the frequency $\omega(\theta_0)$.]

b) Calculate the canonical momentum p_θ and Hamilton's equations for the undamped, undriven case. Calculate the maximum velocity v_1 of the pendulum, starting from rest at the top, using $m = 1$ kg and $\ell = 1$ m. Plot trajectories in phase space starting from equilibrium $\theta = 0$ with initial velocity $v_k = kv_1$ for $k = 0, \pm\frac{1}{5}, \pm\frac{2}{5}, \frac{3}{5}, \ldots 2$. You can obtain a more complete flow of phase space by starting from more equilibrium points: $\theta_0 = 0, \pm 2\pi, \pm 4\pi$.

c) Recalculate the trajectories, adding a frictional force $F = -bv$, starting with $b = 1$ kg/s, normalized to $2\beta = b/m$. Determine the critical damping coefficient b_c, above which oscillations entirely disappear. Repeat the phase space plots for $b = b_c/2, b_c$, and $2b_c$.

d) Recalculate the trajectories, adding a driving force $F = F_0 \cos(\omega t)$ on resonance, $\omega = \omega_0$, with $F_0 = \gamma m g$, and $\gamma = 1$ for this problem. Repeat the trajectories $\theta(t)$ and phase space diagrams $[p_\theta, \theta](t)$ for the same values of b as part c).