University of Kentucky, Physics 404G
Homework #9, Rev. A, due Tuesday, 2019-11-26

1. Despite what we know about earth's outer core being molten (it doesn't propagate earthquake shear waves), the company **Intraterrestrial Express** developed a shady business model to deliver packages 'across' the globe by dropping them through an evacuated tunnel through the center of the earth from southern Argentina to Mongolia. Not many countries are diametrically opposite each other—about the only other options are New Zealand to Spain or Indonesia to Brazil! Ignoring these minor technicalities, we will calculate how long would it take a package dropped from rest to reach the other side, ignoring friction and rotation of the earth.

a) Show that the universal law of gravitation $\boldsymbol{F} = -GMm\hat{\boldsymbol{r}}/r^2$ satisfies Gauss' law just as the electrostatic force $\boldsymbol{F} = Qq\hat{\boldsymbol{r}}/4\pi\epsilon_0 r^2$ does. Use Gauss' law to show that the force inside the earth is springy: $F(r) = -kr$, and determine the effective spring constant of the earth. Integrate $\boldsymbol{F}(r)$ to determine the potential $V(r) = -\int_0^r \boldsymbol{F}(r) \cdot \boldsymbol{dr}$.

b) Using conservation of energy $\epsilon = \frac{1}{2}mv^2 + V(r)$, integrate $dt = dr/v$ to obtain the ETA Δt of the package. Invert the general expression $t(r)$ to obtain the trajectory $r(t)$.

2. Orient Express An imaginary nation practices an esoteric method of interrogation by tying the suspect to a swivel chair, placed a distance a away from a straight infinitely long train track. The angle $\phi(t)$ of the chair is turned to always face an oncoming train, approaching at constant velocity v along the tracks. The suspect has blinders and can only see the train, which appears directly ahead at a distance $\rho(t)$ away. This interrogation method only works as the train approaches from afar, because as it reaches the point of closest approach, it appears to the suspect that the train is mysteriously repelled away by an imaginary force, spoiling any pretense of suspence. Your job as the suspect is to determine from $\rho(t)$ whether or not the train poses an emminent threat, and whether to tell them everything you know or sit firm.

a) Use the Lagrangian for a free particle to obtain the equations of motion for $\rho(t)$ and $\phi(t)$.

b) Use the first integrals from conservation of angular momentum $\ell = mr^2\dot{\phi}$ and energy $\epsilon = \frac{1}{2}mv^2$ to reduce the equations of motion to first order. Substitute $\dot{\phi}$ in the radial equation to obtain a first order differential equation ρ independent of ϕ.

c) Separate variables and integrate to obtain $t(\rho)$. Show that the resulting $\rho(t)$ is a solution of the second order equation of motion for ρ.

d) Substitute $\rho(t)$ into the first order ϕ equation and integrate to obtain $\phi(t)$. Show that $x(t) = \rho\cos\phi = a$ and $y(t) = \rho\sin\phi = vt$, as expected.

3. After recovering from most certain bankruptcy, the newly organized company **Extraterrestrial Express** settled on a more practical, but nonetheless shady, technique to deliver 'packages' in the cargo hold of an ICBM. Follow the same steps as in problem #2, but now with a potential $V(r)$, to calcualte $\rho(t)$ and therefore $\phi(t)$. This gives the formulas for Keplerian motion. In the limit of a near-earth orbit, how long would it take to deliver the same package as in problem #1? [bonus:] What combinations of initial angle and velocity should the company avoid at all cost?

4. [bonus] Each galaxy is immersed in a spherical **dark matter halo** of uniform density, which affects the rotation curves of luminous matter in the galaxy. Using the force $F(r) = -kr$ of #1, calculate the trajectory $\rho(t)$ and $\phi(t)$ of our Sun in the Milky Way, as a general solution.