University of Kentucky, Physics 404G
Homework #10, Rev. B, due Tuesday, 2019-12-10

1. Cross sections

a) Perform a scattering experiment to measure the cross-sectional area σ of each of T identical targets drawn randomly on a chalkboard. Throw as many racquetballs as you can and record the number H of hits versus M of misses. What other measurements do you need to extract the cross section? Calculate the total cross section σ and its statistical uncertainty from your data and compare to the actual area of the target. Note that the cross section includes information about both the target and the scatterer—what area are we actually measuring experimentally? How could you generalize this experiment to measure a differential cross section?

b) Calculate the Rutherford cross section of α particles ($Z_2 = 2$) scattering from the nucleus of gold atoms ($Z_1 = 79$) in a thin foil due to the electric force $F = Z_1 Z_2 e^2/4\pi\epsilon_0 r^2$. This inverse square law has the same form as gravity, and the trajectories have the same paths as Kepler motion (H09 #3) with positive total energy $E > 0$. The repulsive potential corresponds to the opposite branch of the hyperbola than the attractive potential.

c) A rainbow is formed by rays of sunlight refracting into raindrops, internally reflecting, and refracting back out toward the viewer. Calculate the scattering angle θ as a function of the impact parameter s for a given index of refraction n of the raindrop. Calculate the rainbow angle θ_0 in the usual manner as the *caustic* where $d\theta/ds|_{\theta_0} = 0$. [bonus: Calculate the differential cross section, and show it peaks near θ_0.]

[bonus: The mechanical analog of this wavelike phenomenon is a popular model in nuclear scattering called the *optical potential* $V(r) = -V_0\,\theta(a - r)$: a constant potential $V = -V_0$ inside the sphere $r < a$ and a free particle $V = 0$ outside. The step function $\theta(a - r)$ across the surface of the sphere causes a finite impulse which decreases the velocity of the particle as it enters the sphere. What is the relation between the index of refraction n of a light ray, and the ratio E/V_0 of total energy to potential well depth for a particle of mass m, to have the same bend angle into the sphere as the corresponding refraction?]

2. Nuclear Magnetic Resonance (NMR) is a high-precision spectroscopy technique developed in 1946 by Edward Purcell and Felix Bloch. This technology is used in Magnetic Resonance Imaging (MRI) machines. In NMR, the spin $\boldsymbol{s}$ of an atomic nucleus precesses about a constant magnetic field B_0 similar the precession of a spinning top. This characteristic resonance called the *Larmor frequency* $\omega_L = \gamma B$ is proportional to B_0 and to the gyromagnetic ratio γ of the magnetic moment $\boldsymbol{\mu}$ to the angular momentum $\boldsymbol{s}$ of the nucleus, ie. $\boldsymbol{\mu} = \gamma \boldsymbol{s}$.

a) Show that $\gamma = e/2m$ for a charged point particle in a circular orbit, independent of radius r. This does not hold for a composite particle; we define its *g-factor* by $\gamma = g \cdot e/2m$, ie. how much larger than for a point particle of the same mass and charge. A pointlike quantum mechanical particle with spin $s = \hbar/2$ has $g = 2$. For example, the electron has $g_e = 2.002319$, which is slightly larger than 2 due to vacuum polarization. The neutron also has spin $s = \hbar/2$, but $g_n = -3.826$ due to its internal quark structure. Thus its magnetic moment is $\mu_n = g(e/2m)(\hbar/2) = (g/2)\mu_N = -1.91\,\mu_N$. The unit of magnetic moment used in nuclear physics is the *nuclear magneton* $\mu_N = e\hbar/2m_p$, which equals the magnetic moment of a pointlike proton with orbital angular momentum $\ell = \hbar$ (p-orbital).

b) Solve the classical equation of motion (the Bloch equation)

$$\frac{d\boldsymbol{s}}{dt} = \boldsymbol{\tau} = \boldsymbol{\mu}_n \times \boldsymbol{B} = \gamma_n \boldsymbol{s} \times \boldsymbol{B} \tag{1}$$

for the *Larmor precession* of a neutron in a constant magnetic field $\hat{\boldsymbol{z}}B_0$, with initial spin $\boldsymbol{s}_0$ at $t = 0$. Compare this with the precession of a spinning top.

c) The mechanical equations of motion are simplified in a rotating reference frame, where $(\,\hat{\boldsymbol{x}}' \quad \hat{\boldsymbol{y}}'\,) = (\,\hat{\boldsymbol{x}} \quad \hat{\boldsymbol{y}}\,)\begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix}$. [Rabi, Ramsey, Schwinger, Rev. Mod. Phy. **26**, 167, 1954]. Since the operator $\boldsymbol{\omega} dt \times$ generates this rotation, the time derivative becomes $\frac{d\boldsymbol{s}}{dt} = \frac{d\boldsymbol{s}'}{dt} + \boldsymbol{\omega} \times \boldsymbol{s}$, where $\frac{d\boldsymbol{s}'}{dt}$ is with respect to components in the rotating frame. Substitute $\frac{d\boldsymbol{s}}{dt}$ into Eq. 1 and show that the form remains the same except for the replacement of $\boldsymbol{B}$ with the effective field $\boldsymbol{B}' = \boldsymbol{B} + \boldsymbol{\omega}/\gamma_n$. Note this field is zero if the frame is rotating at the Larmor frequency $\boldsymbol{\omega}_L = -\gamma_n \boldsymbol{B}$, and thus the spin remains constant $\boldsymbol{s}' = \boldsymbol{s}_0$. Reconcile this picture with the solution in the static frame.

d) The z-component of spin s_z does not change in a constant magnetic field $\hat{\boldsymbol{z}}B_0$ (called a *holding field* because it preserves the spin state). To transition the spin from up to down, we must use an oscillatory (RF) field $B_1(\hat{\boldsymbol{x}} \cos\omega t + \hat{\boldsymbol{y}} \sin\omega t)$. In the rotating frame with angular velocity $\hat{\boldsymbol{z}}\omega$, show that the total field is $\boldsymbol{B}' = \hat{\boldsymbol{z}}'(B_0 + \omega/\gamma_n) + \hat{\boldsymbol{x}}' B_1$, which is constant. Let θ be the angle between $\boldsymbol{B}'$ and $\hat{\boldsymbol{z}}$, and let the initial spin be $\boldsymbol{s}_0 = \hat{\boldsymbol{z}}$. Show that the z-component of the spin varies as $s_z(t) = s_0(\cos^2\theta + \sin^2\theta \cos\gamma_n B' t) = s_0(1 - 2\sin^2\theta \sin^2(\gamma_n B' t/2))$, which oscillates at the *Rabi flopping frequency* $\omega_R = \gamma_n B'$. Plot the amplitude of oscillation as a function of the RF frequency ω and note the resonance at $\omega = \omega_L$. [bonus: Plot the 3-d trajectory of $\boldsymbol{s}$ in the lab frame over half a Rabi cycle.]