EX1 Solution

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#1 a)

$$x = \xi + d s_\psi \qquad \vec{b}_\xi = \partial_\xi \vec{r} = \hat{x} + 2\xi \hat{y}$$

$$y = \xi^2 - d c_\psi$$

$$\vec{r} = \hat{x} x + \hat{y} y \qquad \vec{b}_\psi = \partial_\psi \vec{r} = \hat{x} d c_\psi + \hat{y} d s_\psi$$

$$g_{\xi\xi} = \vec{b}_\xi \cdot \vec{b}_\xi = 1 + 4\xi^2 \qquad g_{\xi\psi} = d c_\psi + 2\xi d s_\psi \qquad g_{\psi\psi} = d^2$$

$$\Gamma_{\psi\xi\psi} = \vec{b}_\psi \cdot \partial_\xi \vec{b}_\psi = (\hat{x} d c_\psi + \hat{y} d s_\psi) \cdot \vec{0} = \boxed{0}$$

b)

$$\mathcal{L} = \tfrac{1}{2} m\left((1+4\xi^2)\dot{\xi}^2 + 2d(c_\psi + 2\xi s_\psi)\dot{\xi}\dot{\psi} + d^2\dot{\psi}^2\right) - mg(\xi^2 - d c_\psi)$$

$$p_\xi = \mathcal{L}_{,\dot{\xi}} = m\left((1+4\xi^2)\dot{\xi} + d(c_\psi + 2\xi s_\psi)\dot{\psi}\right) = m g_{\xi k} \dot{q}^k$$

$$p_\psi = \mathcal{L}_{,\dot{\psi}} = m\left(d(c_\psi + 2\xi s_\psi)\dot{\xi} + d^2\dot{\psi}\right) = m g_{\psi k} \dot{q}^k$$

$$-\frac{\delta}{\delta\xi}\mathcal{L} = \dot{p}_\xi - \frac{\partial \mathcal{L}}{\partial \xi} = m\Big(8\xi\dot{\xi}^2 + (1+4\xi^2)\ddot{\xi} + d(-s_\psi\dot{\psi} + \cancel{2\dot{\xi} s_\psi} + 2\xi c_\psi \dot{\psi})\dot{\psi} + d(c_\psi + 2\xi s_\psi)\ddot{\psi}$$

$$-4\xi\dot{\xi}^2 - \cancel{2d s_\psi \dot{\xi}\dot{\psi}} + 2g\xi\Big) = 0$$

$$-\frac{\delta}{\delta\psi}\mathcal{L} = \dot{p}_\psi - \frac{\partial \mathcal{L}}{\partial \psi} = m\Big(d(\cancel{-s_\psi\dot{\psi}} + 2\dot{\xi} s_\psi + \cancel{2\xi c_\psi\dot{\psi}})\dot{\xi} + d(c_\psi + 2\xi s_\psi)\ddot{\xi} + d^2\ddot{\psi}$$

$$-d(\cancel{-s_\psi} + \cancel{2\xi c_\psi})\dot{\xi}\dot{\psi} + g d s_\psi\Big) = 0$$

$$\boxed{\begin{aligned} &(1+4\xi^2)\ddot{\xi} + d(c_\psi + 2\xi s_\psi)\ddot{\psi} + 4\xi\dot{\xi}^2 + d(-s_\psi + 2\xi c_\psi)\dot{\psi}^2 + 2g\xi = 0 \\ &\underbrace{d(c_\psi + 2\xi s_\psi)\ddot{\xi} + d^2\ddot{\psi}}_{g_{kj}\ddot{q}^j} + \underbrace{2d s_\psi \dot{\xi}^2}_{\Gamma_{kij}\dot{q}^i\dot{q}^j} + \underbrace{g d s_\psi}_{V_{,k}/m} = 0 \end{aligned}}$$

SHORTCUT:

$$\mathcal{L} = \tfrac{1}{2} m g_{ij}\dot{q}^i\dot{q}^j - V(q) \qquad p_k = \mathcal{L}_{,\dot{q}^k} = m g_{kj}\dot{q}^j$$

$$\dot{p}_k - \frac{\partial \mathcal{L}}{\partial q^k} = \left(m g_{kj}\ddot{q}^j + m g_{kj,i}\dot{q}^j\dot{q}^i\right) - \left(\tfrac{1}{2} m g_{ij,k}\dot{q}^i\dot{q}^j - V_{,k}\right)$$

$$= m\left(g_{kj}\ddot{q}^j + \Gamma_{kji}\dot{q}^j\dot{q}^i\right) + V_{,k} = 0 \quad \text{using} \quad \Gamma_{kji} = \vec{b}_k \cdot \vec{r}_{ji}$$

where $\vec{r}_{\xi\xi} = \vec{b}_{\xi,\xi} = 2\hat{y} \qquad \vec{r}_{\xi\psi} = \vec{0} \qquad \vec{r}_{\psi\psi} = \vec{b}_{\psi,\psi} = \hat{x} d s_\psi - \hat{y} d c_\psi$

c) $\mathcal{L} = \frac{1}{2} m g_{ij} \dot{q}^i \dot{q}^j - V(q)$ $\quad p_k = \mathcal{L}_{,\dot{q}^k} = m g_{kj} \dot{q}^j$ $\quad \dot{q}^j = p_k g^{kj}/m$

$$\mathcal{H} = p_k \dot{q}^k - \mathcal{L} = \frac{1}{2} m g_{kj} \dot{q}^j \dot{q}^k + V(q) = \frac{1}{2m} g^{ij} p_i p_j + V(q)$$

$$g_{ij} \sim \begin{pmatrix} 1+4\xi^2 & d(c_\psi + 2\xi s_\psi) \\ d(c_\psi + 2\xi s_\psi) & d^2 \end{pmatrix} \qquad g^{ij} \sim \frac{\begin{pmatrix} d^2 & -d(c_\psi + 2\xi s_\psi) \\ -d(c_\psi+2\xi s_\psi) & 1+4\xi^2 \end{pmatrix}}{\left((1+4\xi^2) - (c_\psi + 2\xi s_\psi)^2\right) d^2}$$

$$\mathcal{H} = \frac{1}{2m} g^{ij} p_i p_j + V(q)$$

$$\dot{p}_k = -\mathcal{H}_{,q^k} = \frac{-1}{2m}\left(g^{ij}{}_{,k}\, p_i p_j\right) - V_{,k}$$

$$\dot{q}_k = \mathcal{H}_{,p^k} = \frac{1}{m} g^{kj} p_j$$

(must evaluate partials)

#2a) $m\vec{a} = q(\vec{E} + \vec{v} \times \vec{B})$ $\quad \vec{a} = \frac{q\vec{E}}{m} - \hat{z} \times \frac{qB}{m}\vec{v}$ $\quad$ let $\xi = x + iy$ $\quad \eta = \dot{\xi} = v_x + i v_y$

[you can also integrate this directly in Matlab]

$$\dot{\eta} = \varepsilon - i\omega_c \eta = -i\omega_c(\eta - \eta_d) \qquad \varepsilon = \frac{qE}{m} \quad \omega_c = \frac{qB_z}{m} \quad \eta_d = \frac{-i(E_x + iE_y)}{B_z}$$

$$(\eta - \eta_d) = (\eta_0 - \eta_d)\, e^{-i\omega_c t} \qquad \xi = \xi_0 + \eta_d t + \frac{\eta_0 - \eta_d}{-i\omega_c}\left(e^{-i\omega_c t} - 1\right)$$

$$\xi_0 = 0 \quad \eta_0 = 0 \quad \eta_d = -i\,E/B = -i\,\frac{10\,V/m}{10\,\mu T} = -i \cdot 10^6\,m/s \quad \omega_c = \frac{qB}{m} = \frac{(1.6\times10^{-19}C)(10\,\mu T)}{(9.1\times10^{-31}kg)} = 1.76\times10^6/s \quad t = 1\,\mu s$$

$$\xi(1\mu s) = \eta_d t - i\frac{\eta_d}{\omega_c}\left(e^{i\omega_c t} - 1\right) = 0.675 - i\,0.441 = x + iy\ [m]$$

In Matlab, you could use $\varepsilon = \frac{(1.6\times10^{-19}C)(10\,V/m)}{(9.1\times10^{-31}kg)} = 1.76\times10^{12}\,m/s^2$ instead of η_d.

```
octave> ed=1.76e12; wc=1.75e6; t1=1e-6; hd=-1e6i;
octave> [t,u]=ode45( @(t,u)[u(2); ed-1i*wc*u(2)], [0,t1], [0;0]); x0=u(end,1)
  =>   6.7713e-01 - 4.4022e-01i
```

b) repeat for $(E = 10.01\ V/m,\ B = 10\,\mu T)$ and $(10\,V/m,\ 10.01\,\mu T)$

```
octave> ed1=1.76e12*1.001; wc1=1.75e6*1.001;
octave> [t,u]=ode45( @(t,u)[u(2);ed1-1i*wc*u(2)], [0,t1], [0;0])
octave> x1=u(end,1)  % increase E by .1%
x1 =  0.67781 - 0.44066i
octave> [t,u]=ode45( @(t,u)[u(2);ed-1i*wc1*u(2)], [0,t1], [0;0]);
octave> x2=u(end,1) % increase B by .1%
x2 =  0.67677 - 0.44053i
octave> J=([ real([x1,x2]); imag([x1,x2]) ]-[ real(x0); imag(x0) ])/0.001
J =  0.67713  -0.36476
    -0.44022  -0.30435
```

```
octave> xt=0.5-0.5i     %target endpoint
xt =  0.50000 - 0.50000i
octave> xt-x0
ans = -0.177131 - 0.059776i
octave> dx=[real(xt-x0);imag(xt-x0)]  %desired change
dx =  -0.177131
      -0.059776
octave> dEB=J\dx   %fractional increase in E,B
dEB =  -0.087563         E1= 10-.87563= 9.12437 V/m
        0.323062         B1= 10+3.23062=13.23062 uT
octave> ed3=1.76e12*(1+dEB(1)); wc3=1.75e6*(1+dEB(2));
octave> [t,u]=ode45( @(t,u)[u(2);ed3-1i*wc3*u(2)], [0,t1], [0;0]);
octave> x3=u(end,1)
x3 =  0.50255 - 0.47329i  %very close to target!
```