University of Kentucky, Physics 404G
Homework #3, Rev. A, due Tuesday, 2020-09-15

1. The **Cartesian, cylindrical**, and **spherical** coordinate systems are by far the most common. Do parts a),b),c), and e) for each of these coordinate systems, d) for cylindrical only:

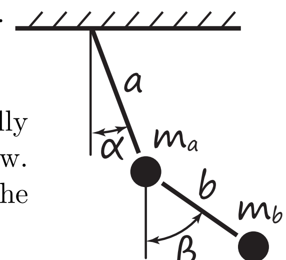
a) Determine the coordinate transformation functions $q^i(\boldsymbol{r})$ in terms of each of the other two systems. *Hint: combine the simpler Cartesian-cylindrical and cylindrical-spherical transformations to obtain Cartesian-spherical.*

b) Pick a point $\boldsymbol{r}_0$ and illustrate the three coordinate isosurfaces $q^i(\boldsymbol{r}) = q^i_0$ (constant) that pass through $\boldsymbol{r}_0$, labeling all lengths and angles in your diagram. For each coordinate q^i, identify the curve $\boldsymbol{s}(q^i; q^j_0, q^k_0)$ at the intersection of two surfaces of constant $q^j = q^j_0$ and $q^k = q^k_0$.

c) Calculate $\boldsymbol{b}_i = \partial\boldsymbol{s}/\partial q^i$ using $\boldsymbol{ds} = \hat{\boldsymbol{x}}dx + \hat{\boldsymbol{y}}dy + \hat{\boldsymbol{z}}dz$. Normalize $\boldsymbol{b}_i = \hat{\boldsymbol{e}}_i h_i$ to find the *unit vectors* and *line element* $\boldsymbol{ds} = \hat{\boldsymbol{e}}_i h_i dq^i$, [bonus: *area element* $\boldsymbol{da} = \frac{1}{2}\boldsymbol{ds} \times \boldsymbol{ds} = \hat{\boldsymbol{e}}_k\, h_i h_j\, dq^i dq^j$ for i, j, k cyclic], and [bonus: *volume element* $d\tau = \frac{1}{3}\boldsymbol{ds} \cdot \boldsymbol{da} = h_1 h_2 h_3\, dq^1 dq^2 dq^3$ (for orthogonal coordinates)]. Note that the scale factors h_θ and h_ϕ for angular coordinates are just radius of curvature, according to the arc length formulae $ds_\theta = r d\theta$ and $ds_\phi = \rho d\phi$.

d) Invert cylindrical coordinates to obtain $(\rho, \phi, z) = f^{-1}(x, y, z)$. Calculate the covariant basis $\boldsymbol{b}^i = \nabla q^i = \hat{\boldsymbol{e}}_i/h_i$ to verify $\boldsymbol{b}_i \cdot \boldsymbol{b}^j = \delta^j_i$. [bonus: Same for spherical coordinates].

e) Calculate $g_{ij} = \boldsymbol{b}_i \cdot \boldsymbol{b}_j = \mathrm{diag}(h_1^2, h_2^2, h_3^2)$ and [bonus: $\boldsymbol{\Gamma}_{ij} = \partial\boldsymbol{b}_i/\partial q^j$].



2. The position of the lower mass m_b of a **double pendulum** is most naturally parametrized by the non-orthogonal coordinate system (α, β), illustrated below. The position of the upper mass m_a is trivially parametrized by α, but that of the lower mass m_b depends on both angles (α, β).

a) Calculate and draw the α and β coordinate lines, the contravariant basis vectors $\boldsymbol{b}_i$ [bonus: and covariant basis $\boldsymbol{b}^i$]. Use the components of $\boldsymbol{ds}$ to obtain the velocity $\boldsymbol{v} = \dot{\boldsymbol{r}} = \boldsymbol{b}_i \dot{q}^i$ of the lower mass.

b) Calculate the metric $g_{ij} = \boldsymbol{b}_i \cdot \boldsymbol{b}_j$ and thus $ds^2 = g_{ij} dq^i dq^j$ to obtain the kinetic energy $T_b = \frac{1}{2} m_b v^2 = \frac{1}{2} m_b g_{ij} \dot{q}^i \dot{q}^j$ of the lower mass. Raise the indices of $\boldsymbol{b}_i$ with the inverse metric g^{ij} to obtain the covariant basis $\boldsymbol{b}^i$. [bonus: Compare with the direct calculation of $\boldsymbol{b}^i = \nabla q^i$ from a)].

c) Calculate the connection vectors $\boldsymbol{\Gamma}_{ij} = \partial\boldsymbol{b}_i/\partial q^j$. Compare the covariant components $\Gamma_{ijk} = \boldsymbol{\Gamma}_{ij} \cdot \boldsymbol{b}_k$ of $\boldsymbol{\Gamma}_{\alpha\alpha}$ with those obtained directly from the metric $\Gamma_{ijk} = \frac{1}{2}(g_{ik,j} + g_{jk,i} - g_{ij,k})$. Use Γ_{ijk} to obtain the covariant components of acceleration $\boldsymbol{A} = \dot{\boldsymbol{v}} = \boldsymbol{b}_k \ddot{q}^k + \boldsymbol{\Gamma}_{ij} \dot{q}^i \dot{q}^j$ of the mass m_b.

d) Calculate the potential of gravity $V(\alpha, \beta) = mgh$ for both masses m_a and m_b in terms of (α, β) to construct the Lagrangian $\mathcal{L} = T - V$. Compare the two equations of motion for α and β, in the covariant components of $\boldsymbol{F} = m\boldsymbol{A}$ using $F_i = \partial V/\partial q^i$ and A_i from above, with the direct calculation using Lagrange's equations.

e) Computer exercise: numerically integrate the four first order equations of motion in the variables $\boldsymbol{q} = (\alpha, \beta, \dot{\alpha}, \dot{\beta})$ for $a = 2$ m and $b = 1$ m, $m_a = m_b = 1$ kg, starting from initial conditions $\boldsymbol{q}_0 = (.1,\ .2,\ 0,\ 0)$, in units of [rad] and [rad/s]. Vary the initial value β_0 to discover two *modes* of pure periodic motion, and calculate their [eigen]frequencies. Increase α_0 and β_0 until the motion becomes *chaotic* (unpredictable). *Hint: start with the code for this simple pendulum.*