University of Kentucky, Physics 404G
Homework #4, Rev. A, due Thursday, 2020-09-22

1. Calculus of variations was foreseen by Pierre de Fermat, who realized that *Snell's law* could be viewed as a *principle of least time* (or *optical path length*) for a ray of light to travel between two points. He essentially invented *calculus* to solve this and other types of minimization problems, although he mislabeled the use of his infinitesimal ϵ as *adequality* instead of more correctly as $\lim_{\epsilon\to 0}$. The term "adequality" for "approximate equality" was borrowed from Diophantus, in whose book he penciled *Fermat's last theorem* in the margin without proof.

Maupertuis, who felt that "Nature is thrifty in all of its actions," was bothered that time and distance were not treated on an equal footing. He developed the principle of **least action**, minimizing instead the integral $\int v ds = \int 2T/m\, dt$, one of the first appearances of *vis viva*. This was further expanded from optics to mechanics by Euler as the *abbreviated action* $S_0 = \int p\, dq$. The modern action $S = \int \mathcal{L} dt$ and associated equations of stationary action $(\frac{d}{dt}\frac{\partial}{\partial \dot{q}} - \frac{\partial}{\partial q})\mathcal{L} = 0$ were developed by Lagrange and Hamilton. This same principle extends to quantum field theory and *Feynman path integrals*. We have already seen that these equations are just Newton's law written in covariant components. This problem explores the interpretation of the expression in parenthesis as a variational derivative $\frac{\delta}{\delta q}$ for all path variations δq.

a) Minimize the time taken for a ray of light to travel from $P_1 = (x_1, y_1)$ to $P_2 = (x_2, y_2)$ through an interface at $y = 0$ between two materials of *index of refraction* n_1 and n_2. Assume for now that the shortest path between two points is a straight line, so that the only variable is x, where the ray crosses the interface at $y = 0$. Derive Snell's law from your answer. [bonus: extend your solution to the case of a ray crossing multiple parallel interfaces, and show it is equivalent to $\nabla f(x_1, x_2, \ldots) = 0$ for crossing points x_i.]

b) Use Fermat's *principle of stationary optical path length* to calculate the image of a mirage by minimizing the functional $I[x(y)] = \int n ds = \int n(y)\sqrt{1 + x'(y)^2} dy$, or setting its *functional derivative* $\frac{\delta I}{\delta x} \equiv (-\frac{d}{dy}\frac{\partial}{\partial x'} + \frac{\partial}{\partial x})n(y)\sqrt{1 + x'(y)^2}$ to zero for the path $y(x)$. [Nearing, p. 520] Show the differential form of Snell's law, that $n(y)\frac{x'(y)}{1+x'(y)^2} = n\frac{dx}{ds} = n\sin(\theta) = C$ is constant along the path or, rearranging, that $x'(y) = C/\sqrt{n(y)^2 - C^2}$. To solve this we need to know the index of refraction $n(y)$. Let us assume it is linear with height, $n(y) = n_0 + \alpha y$. Use the substitution $n(y) = C\cosh(\theta)$ to integrate $x'(y)$. Invert the solution of $x(y)$ to obtain $y = \frac{1}{\alpha}(-n_0 + C\cosh(\frac{\alpha}{C}(x - x_0)))$. Note that a chain fixed at both ends has the same shape for a similar reason: it also hangs under a linear potential, $V(y) = V_0 + mgy$, of gravity in this case.

2. Phase space of a Harmonic Oscillator—All of our computer simulations have converted second order ODEs involving $\ddot{x}$ into first order equations in $u = (x, v)$. This is formalized in a more symmetric way by Hamilton's equations of the canonical conjugate variables $\boldsymbol{a} = (x, p)$, as discussed in class.

a) The Lagrangian for the harmonic oscillator is $\mathcal{L} = T - V = \frac{1}{2}mv^2 - \frac{1}{2}kx^2$, where $v = \dot{x}$. Calculate the canonical momentum $p = \partial\mathcal{L}/\partial v$, and solve Lagrange's equation of motion on x.

b) Calculate $\mathcal{H}(p, q) = p\dot{q} - \mathcal{L}(\dot{q}, q)$, obtain Hamilton's equations for this Hamiltonian. Solve them by treating (x, p) as a complex number $a = \frac{1}{\sqrt{2}}(x\sqrt{k} + ip/\sqrt{m})$ and matching its time derivative with Hamilton's equations.

c) Plot the trajectory a in the complex plane. Show that $\mathcal{H} = a^*a$. Note that the polar representation $|a|e^{i\phi}$ is called *action-angle* coordinates, an alternative to (x,p) with trivial equations of motion. In quantum mechanics a is made dimensionless by factoring out $\sqrt{\hbar\omega}$ so that $\mathcal{H} = \hbar\omega(a^\dagger a + \frac{1}{2})$.

d) Integrate the abbreviate action $S_0 = \int p\, dx$ around one cycle. Note that a circle is the geometric shape with the maximum area, as demanded by Hamilton's *principle of stationary action.*

3. [bonus] Physical interpretation of the **vector potential** $\boldsymbol{A}$. The quantity $q\boldsymbol{A}$ has physical significance as potential momentum, just as qV is potential energy. The purpose of this problem is to explore the implications of this connection. For more background information see Am. J. Phys. **64**, 1368 (1996).

a) Show that $\partial\rho/\partial t + \nabla\cdot\boldsymbol{J} = 0$ (conservation of charge). In 4-vector notation, this is written $\partial_\mu J^\mu = 0$, where $\partial = (\partial_t, \nabla)$, and $J = (\rho, \boldsymbol{J})$.

b) For a continuous distribution $\rho(\boldsymbol{r},t)$ of charges in a velocity field $\boldsymbol{v}(\boldsymbol{r},t)$ so that $\boldsymbol{J} = \rho\boldsymbol{v}$, show that the *convective derivative* of ρ (the time derivative of $\rho(\boldsymbol{r}(t),t)$ along the path $\boldsymbol{r}(t)$ with velocity $\boldsymbol{v} = d\boldsymbol{r}/dt$) is $d\rho/dt = \partial\rho/\partial t + \boldsymbol{v}\cdot\nabla\rho$, where $\partial\rho/\partial t$ is the time derivative of ρ at a fixed position $\boldsymbol{r}$ and $\nabla\rho$ is the gradient of ρ at a fixed time. Use part a) to show that $d\rho/dt + \rho\nabla\cdot\boldsymbol{v} = 0$. What is the physical interpretation of $\nabla\cdot\boldsymbol{v}$ for a velocity field?

c) Show that using the velocity-dependent potential $U = q(V - \boldsymbol{v}\cdot\boldsymbol{A})$ in Lagrange's equation produces the Lorentz force:

$$\boldsymbol{F} \equiv \frac{d}{dt}\frac{\partial U}{\partial \boldsymbol{v}} - \frac{\partial U}{\partial \boldsymbol{r}} = q(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}), \tag{1}$$

where $\partial/\partial\boldsymbol{r} = \nabla = \hat{\boldsymbol{x}}\partial_x + \hat{\boldsymbol{y}}\partial_y + \hat{\boldsymbol{z}}\partial_z$ and $\partial/\partial\boldsymbol{v} = \nabla_{\boldsymbol{v}} = \hat{\boldsymbol{x}}\partial_{v_x} + \hat{\boldsymbol{y}}\partial_{v_y} + \hat{\boldsymbol{z}}\partial_{v_z}$ are gradients with respect to the independent variables $\boldsymbol{r}$ and $\boldsymbol{v}$, respectively. Use $\boldsymbol{E} \equiv -\nabla V - \partial\boldsymbol{A}/\partial t$, which includes time dependence. This generalizes the standard formula $\boldsymbol{F} = -\partial U/\partial\boldsymbol{r} = -\nabla U$. Thus the electromagnetic Lagrangian is $L = T - U = \frac{1}{2}mv^2 - q(V - \boldsymbol{v}\cdot\boldsymbol{A})$.

d) Calculate the *canonical momentum* $\boldsymbol{\pi} \equiv \frac{\partial L}{\partial \boldsymbol{v}}$. This generalized momentum includes contributions from both the particle and the magnetic field.

e) Calculate the *Hamiltonian* $H \equiv \boldsymbol{\pi}\cdot\boldsymbol{v} - L$. Express H in terms of $\boldsymbol{\pi}$ and $\boldsymbol{r}$, not $\boldsymbol{v}$. This is the total energy in the system. Thus the Hamiltonian and canonical momentum $(H, \boldsymbol{\pi})$ are the generalized energy and momentum of a system. Each includes a kinetic and potential term.

f) Show that $\frac{dH}{dt} = \frac{d}{dt}(T + qV) = \frac{\partial}{\partial t}q(V - \boldsymbol{v}\cdot\boldsymbol{A}) = \frac{\partial U}{\partial t}$, where $\frac{\partial}{\partial t}$ acts on the potentials, but not $\boldsymbol{r}$ or $\boldsymbol{v}$. Thus the total energy H is conserved if the generalized potential is time-independent.

g) Show that $\frac{d\boldsymbol{\pi}}{dt} = \frac{d}{dt}(\boldsymbol{p} + q\boldsymbol{A}) = -\frac{\partial}{\partial \boldsymbol{r}}q(V - \boldsymbol{v}\cdot\boldsymbol{A}) = -\frac{\partial U}{\partial \boldsymbol{r}}$, where again $\frac{\partial}{\partial \boldsymbol{r}}$ acts on the potentials, but not $\boldsymbol{v}$. Thus $\boldsymbol{\pi}$ is conserved if the generalized potential is position-independent.

h) Show that the electromagnetic fields do not change if $V \to V + \partial\chi/\partial t$ and $\boldsymbol{A} \to \boldsymbol{A} + \boldsymbol{\nabla}\chi$. Calculate the Noether current associated with this *gauge symmetry*.