University of Kentucky, Physics 404G
Homework #5, Rev. A, due Thursday, 2020-10-01

1. It has been claimed that World War II was won by the **Magnetron**, a vacuum device which converts electrical energy into microwave radiation. It enabled short-wavelength tactical radar, which used a frequency too high for electrical circuits of the time. Radiation is emitted from several cylindrical microwave cavities surrounding the central bore of a solid copper waveguide under vacuum. A central *thermionic cathode* of radius a at negative high voltage V emits electrons, which accelerate to the grounded *anode* (inner cylindrical face of the cavity) of radius b. A longitudinal magnetic field $\boldsymbol{B}$ sweeps the electrons in a spiral path around cylindrical void, exciting resonant currents in the outer microwave cavities as they sweep by. The electric and magnetic fields are perpendicular in this *crossed-field device*.

a) [bonus: In L16 we developed the equation of motion for an electron in a magnetic field, $\dot{\eta} = -i\omega\eta$, using complex coordinates $\xi = x + iy$ and $\eta = \dot{\xi} = v_x + iv_y$, where $\omega = qB/m$ is the *cyclotron frequency*. Show that the equation of motion for an electron in constant electric $\boldsymbol{E} = \hat{\boldsymbol{x}}E_0$ and magnetic $\boldsymbol{B} = \hat{\boldsymbol{z}}B_0$ fields is $\dot{\eta} = -i\omega(\eta + iv_d)$, where $v_d = E_0/B_0$. Solve this equation to obtain the *cycloid* motion of the electron in these fields. What is the physical interpretation of iv_d?]

b) Express the force $\boldsymbol{F} = q\boldsymbol{E}$ due to the electric field $\boldsymbol{E} = \hat{\boldsymbol{\rho}}V/[\rho\ln(b/a)]$ in terms of ξ to obtain the equations of motion $\dot{\xi} = \eta$ and $\dot{\eta} = \lambda/\xi^* - i\omega\eta$. What is the value of the constant λ when $a = 0.5$ cm, $b = 2.5$ cm, and $V = -4$ kV is applied to the *cathode*? Integrate the equations in Matlab for an electron emitted essentially at rest from the cathode, and $B = 1$ T. Before you start, what will the motion look like qualitatively? Determine the *Hull cut-off* field B_c, such that an electron barely grazes the surface of the anode and returns to the cathode. [bonus: Plot $B_c(V)$.]

c) The electrons from part a) excite eight cylindrical resonant cavities of radius $r = 1$ cm distributed around the central cavity, each one separated from the anode (at $\rho = b$) by a $w = 4$ mm long, $t = 5$ mm wide channel cut in the solid cylindrical copper block of thickness d. Calculate the resonant frequency using the inductance $L = N^2\mu_0\pi r^2/d$ and capacitance $C = \epsilon_0 wd/t$. Optimize both the voltage V and magnetic field B such that the electrons just graze past the resonators at the resonant frequency. *Hint: use conservation of energy to determine the required voltage and then determine the magnetic field. Warning: Don't try this at home without also including the space-charge effect and its associated "pinwheel"!*

2. Chaotic Damped Driven Pendulum

a) Calculate the Lagrangian $\mathcal{L}$, canonical momentum p_θ, and Hamiltonian $\mathcal{H}$ to obtain Hamilton's equations for an undamped, undriven pendulum of mass $m = 1$ kg and length $\ell = 1$ m. Calculate the maximum velocity v_1 of the pendulum, starting from rest at the top. Plot the trajectories $[p_\theta, \theta](t)$ in phase space starting from equilibrium $\theta_0 = 0$ with initial velocities $v_k = kv_1$ for $k = 0, \pm\frac{1}{5}, \pm\frac{2}{5}, \frac{3}{5}, \ldots 2$. Obtain a complete flow in space by starting from many equilibrium points: $\theta_0 = 0, \pm 2\pi, \pm 4\pi$. [bonus: Calculate the exact trajectory, starting from rest at angle θ_0, in terms of the Jacobi elliptic function $sn(u;m)$ and determine the frequency $\omega(\theta_0)$.]

b) Determine the critical damping coefficient b_c of an additional frictional force $F = -bv$, and compare the trajectories in phase space for $b = b_c/2, b_c$, and $2b_c$.

c) Repeat the damped trajectories in phase space, adding a resonant force $F = \gamma mg\cos(\omega_0 t)$, driven at $\gamma = 1$. [bonus: Determine the minimum value of γ where the motion becomes chaotic.]

3. Discovery of the **Higgs boson** in 2012 confirmed the last piece of the Standard Model of particle physics. But this elusive particle was not actually observed: only the two photons [or other particles] into which it decays after $1/\Gamma \sim 10^{-22}$ s. The observed signature was a resonance in the histogram of the mass or frequency $E = mc^2 = \hbar\omega$ reconstructed from the energy of the two photons, according to the damped oscillations of its quantum mechanical wavefunction $\Psi(t) = \Psi_0 e^{-\Gamma t/2} e^{i\omega_1 t}$, with probability $P(t) = |\psi(t)|^2 = e^{-\Gamma t}$ decaying at the rate Γ.

a) Calculate the initial velocity $v_0 = \Im\dot{\Psi}(0)$ of $\Im\Psi(t) = \Psi_0 e^{-\Gamma t/2} \sin(\omega_1 t)$. Using the Fourier method, treat the system as a damped oscillator driven with the impulse $f_0(t) = v_0 \delta(t)$, acting at $t = 0$ to kick it with the initial velocity, not interfering afterwards. Using the principle of supperposition, formally integrate all components to express the resulting impulse response $\Im\Psi(t)$ as an integral over the amplitude function $A(\omega)$. This shows that the Fourier transform of the free decay wavefunction $\Im\Psi(t)$ directly encodes the resonance curve $A(\omega)$ of the associated driven oscillator. This so-called *Breit-Wigner resonance* relates the spectral linewidth to the decay rate of the wave function, for anything from laser transitions to short-lived high-energy particles.

b) Show that the Full Width at Half Maximum (FWHM) of the resonance in $|A|^2(\omega)$ is $\Gamma = 2\beta$. Estimate the frequency ω_1 and lifetime $1/\Gamma$ of the Higgs boson from the associated resonance in the *missing mass* spectrum in Fig. 4 of the discovery paper. Calculate the Q value of this resonance. Note that Q-values upwards of 10^{11} have been achieved in atomic physics.