University of Kentucky, Physics 404G
Homework #7, Rev. B, due Tuesday, 2020-10-20

1. Gravity waves propagate along the interface between a liquid and a gas (or any two fluids of different density, for example air and water), as gravity or *buoyancy* tries to restore equilibrium. The particles in the liquid follow elliptical trajectories with their amplitude decaying exponentially in depth. If the velocity field $\boldsymbol{v}(x,z)$ is irrotational ($\nabla \times \boldsymbol{v} = 0$), it can be represented by the gradient of a scalar flow potential $\boldsymbol{v} = -\nabla\phi(x,z)$. If the fluid is also incompressible ($\nabla \cdot \boldsymbol{v} = 0$), its flow satisfies the Laplace equation $\nabla^2\phi = 0$. Let the gas-liquid interface be at height $z = \eta(x,t)$ above the equilibrium level at $z = 0$, and h be water depth (the sea bed is at $z = -h$).

In addition to gravity, surface tension γ, exerts a downward pressure $P = -\gamma\nabla_\perp^2\eta$ on the liquid, also propagating ripples on the surface. This is the two-dimensional analog of the vertical force $dF = Tf''dx$ of a wave on an element of string under horizontal tension T. Both of these effects can be described by Airy wave theory, which we develop below.

a) Show that the function $\phi(x,z,t) = A\cosh(k(z+h))\cos(kx-\omega t)$ is a solution of $\nabla^2\phi = 0$. Plot the equipotentials of ϕ at $t = 0$, with arrows showing the direction of $\boldsymbol{v}$.

b) Show this solution satisfies the boundary condition $v_z(x,-h) = 0$, at the bottom of the liquid. The boundary condition on the top surface is $v_z = \dot{\eta}$, evaluated at $z = 0$ (the boundary to 0^{th} order). Find A so that ϕ satisfies this boundary condition at the interface $\eta(x,t) = a\sin(kx-\omega t)$.

c) Integrating Newton's law over z leads to Bernoulli's law $\rho\partial_t\phi = \rho g\eta - \gamma\partial_x^2\eta$, where ρ is the mass density of the liquid. Substitute ϕ and η into Bernoulli's law at $z = 0$ to obtain the dispersion relation $\omega^2 = (gk + \frac{\gamma}{\rho}k^3)\tanh(kh)$. Plot $\omega(k)$, $v_\phi(k) = \omega/k$, and $v_g(k) = d\omega/dk$.

d) Calculate the wavelength λ_c below which waves are dominated by surface tension, using $\gamma = 72.8$ mN/m and $\rho = 1.00$ g/cm^3 for water. What is the dispersion relation in this limit?

e) Approximate $\phi(x,z,t)$ and $\omega(k)$ in the deep water limit, where $kh >> 1$. [bonus: Do individual crests move forward or backward within the wave packet?]

f) Approximate $\omega(k)$ in the shallow water limit, and show that all frequencies have the same velocity. What is the speed of a tsunami ($\lambda \approx 100$ km) in 10 km deep [shallow!] ocean waters? How long will it take one wavelength to pass?

2. [bonus] **Elastic waves** in solids, also known as body, bulk, seismic, stress, or strain waves, are the three-dimensional analog of waves traveling along a Slinky. They have three polarizations: one longitudinal polarized acoustic P-wave, (primary or pressure), and two transverse polarized S-waves (secondary or shear). [Note there are three completely different and inconsistent definitions of S- and P-waves for seismic, optical, and quantum mechanical angular momentum waves!]

Pressure and shear waves are both described in terms of elastic deformation. Similar to the displacement $f(x,t)$ of coils in a Slinky, the displacement field $\boldsymbol{u}(\boldsymbol{r},t)$ describes the shift of the particle at position $\boldsymbol{r}$ in equilibrium to the new position $\boldsymbol{r}+\boldsymbol{u}$. Elastic strain (deformation) $\boldsymbol{\epsilon}$ indicates the change in displacement $\boldsymbol{du} = \boldsymbol{\epsilon}\cdot\boldsymbol{dr}$ between neighboring particles, analogous to $f'(x,t)$ for the Slinky. This symmetric *strain tensor* $\epsilon_{ij} \equiv \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ is the matrix of all nine possible linear deformations. On the other hand, the *stress tensor* $\tau_{ij} = dF_i/da_j$ describes all three components of force dF_i per area along all three independent directions of surface area da_j of the interface between two neighboring elements at $\boldsymbol{r}$ and $\boldsymbol{r}+\boldsymbol{dr}$ at equilibrium in the bulk, so that the equal and opposite force between these elements is $\boldsymbol{dF} = \pm\boldsymbol{\tau}\cdot\boldsymbol{da}$. The normal component of stress to the interface ($i = j$) is called *pressure*, while the tangential components ($i \neq j$) are called *shear*, each propagating their own polarization. Stress transfers energy and momentum across the interface, propagating the wave according to Newton's second law $\boldsymbol{dF} = dm\,\boldsymbol{A}$, which is written $\partial_j\tau_{ij} = \rho\ddot{u}_i$, analogous to $dF = \mu\,dx\ddot{f}$ for the Slinky.

The generalization of Hooke's law, $dF = \kappa f'' dx$, from the Slinky to the bulk of an elastic material, $\tau_{ij} = \lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij}$, relates the stress (force) $\boldsymbol{\tau}$ to the strain (stretch) $\boldsymbol{\epsilon}$. This is the most general isotropic and rotationally invariant relation, which contains two independent "spring constant" *Lamé parameters*: λ (pressure), and μ (sheer). Other combinations of these constants, each with their own physical interpretation are: Poisson's ratio $\nu = \lambda/2(\lambda+\mu)$, the P-wave (longitudinal, pressure) modulus $M = \lambda + 2\mu$, the S-wave (transverse, shear) modulus $G = \mu$, the *bulk modulus* (compressibility) $K = M - \frac{4}{3}G$, and *Young's modulus* (stretchiness) $E = 2G(1+\nu) = 3K(1-2\nu)$. Nonviscous fluids do not support sheer strain ($\mu = 0$) and only accommodate P-waves. For an ideal gas, the isentropic bulk modulus $K_S = \gamma P$ is used, where the *adiabatic index* $\gamma = C_P/C_V$ is the ratio of heat capacities at constant pressure and volume, respectively, and P is the equilibrium pressure. In air at STP, $\gamma = 1.41$ and $P = 1$ bar $= 100$ kPa.

a) Combine Hooke's and Newton's laws in the form given above to obtain the wave equation $\rho\,\ddot{\boldsymbol{u}} = M\,\nabla\nabla\cdot\boldsymbol{u} - G\,\nabla\times\nabla\times\boldsymbol{u}$. Note that the two operators $\nabla_\parallel^2 \equiv \nabla\nabla\cdot$ and $\nabla_\perp^2 \equiv -\nabla\times\nabla\times$ are the longitudinal and transverse projections of the *Laplacian* $\nabla^2 \equiv \nabla\cdot\nabla = \nabla_\parallel^2 + \nabla_\perp^2$ (the fundamental second vector derivative) that describe the curvature of longitudinal and transverse waves, respectively.

b) For a P-wave, $\nabla\times\boldsymbol{u} = 0$. Take the divergence of both sides of the wave equation to show that $\rho\ddot{n} = M\nabla^2 n$, where $n \equiv \nabla\cdot\boldsymbol{u}$ is the compression of the medium. Show that the wave $n = ae^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$ is a solution with amplitude a traveling in the direction $\hat{\boldsymbol{k}}$, and obtain the dispersion relation $\omega(\boldsymbol{k})$. Calculate the velocity of a seismic P-wave ($K = 200$ GPa, $\rho = 1700$ kg/m^3 for rock), and of sound in air ($\rho = 1.225$ kg/m^3), water ($K = 2.2$ GPa), and steel ($K = 160$ Gpa, $\rho = 8050$ kg/m^3).

c) For an S-wave, $\nabla\cdot\boldsymbol{u} = 0$. Take the curl of both sides of the wave equation to show that $\rho\ddot{\boldsymbol{m}} = G\nabla^2\boldsymbol{m}$, where $\boldsymbol{m} \equiv \nabla\times\boldsymbol{u}$ is the sheer strain which is perpendicular to the direction of propagation. Show that the wave $\boldsymbol{m} = \boldsymbol{a}e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$ is a solution with transverse amplitude $\boldsymbol{a}$ traveling perpendicular to $\boldsymbol{k}$, and obtain the dispersion relation $\omega(\boldsymbol{k})$. Calculate the velocity of a seismic S-wave ($M = 100$ GPa for rock), and of a sheer wave in steel (M =79.3 GPa). There are no S-waves in water, air, or in the molten outer core of the earth, which is how we know it exists.