

University of Kentucky, Physics 404G
Homework #2, Rev. A, due Tuesday, 2021-09-14

1. Projections: relational versus parametric geometry.

a) Show graphically that the following equations define a set of points $\{\mathbf{x}\}$ on a **line** or **plane**,

	relational	parametric
line	$\{\mathbf{x} \mid \mathbf{a} \times \mathbf{x} = \mathbf{d}\}$	$\{\mathbf{x} = \mathbf{x}_1 + \mathbf{a}\alpha \mid \alpha \in \mathbb{R}\}$
plane	$\{\mathbf{x} \mid \mathbf{A} \cdot \mathbf{x} = D\}$	$\{\mathbf{x} = \mathbf{x}_2 + \mathbf{b}\beta + \mathbf{c}\gamma \mid \beta, \gamma \in \mathbb{R}\}$

where $\mathbf{a}, \mathbf{d}, \mathbf{A}, D$ are constants which define the geometry, $\mathbf{x}_{1,2}$ are fixed points on the line and plane respectively, and α, β, γ are parameters that vary along the line/plane (they uniquely parametrize points in the line/plane). [bonus:] Show the 5th relation $\{\mathbf{x} = \mathbf{x}_2 + \mathbf{A} \times \boldsymbol{\delta} \mid \boldsymbol{\delta} \in \mathbb{R}^3\}$.

b) What constraint between \mathbf{a} and \mathbf{d} is implicit in the formula $\mathbf{a} \times \mathbf{x} = \mathbf{d}$? Likewise, what relations between \mathbf{b}, \mathbf{c} , and \mathbf{A} must be satisfied if both equations describe the same plane? For the line and the plane, substitute \mathbf{x} from the parametric form into the relational form to show that they are consistent. What are \mathbf{d} and D in terms of \mathbf{a}, \mathbf{x}_1 and \mathbf{A}, \mathbf{x}_2 , respectively?

c) We define $\tilde{\mathbf{a}} \equiv \mathbf{A}/(\mathbf{a} \cdot \mathbf{A})$. It is parallel to \mathbf{A} but *normalized* in the sense that $\mathbf{a} \cdot \tilde{\mathbf{a}} = 1$. Using the BAC-CAB rule, show that $\mathbf{x} = \mathbf{a}(\tilde{\mathbf{a}} \cdot \mathbf{x}) - \tilde{\mathbf{a}} \times (\mathbf{a} \times \mathbf{x})$ for any \mathbf{x} . This is a non-orthogonal projection of \mathbf{x} into a vector parallel to the line and a vector parallel to the plane. Illustrate this projection and show which term corresponds to each.

d) [bonus:] Using c), calculate the point \mathbf{x}_0 at the intersection of the line and plane in terms of $\mathbf{a}, \mathbf{d}, \mathbf{A}, D$. Verify this by showing \mathbf{x}_0 satisfies the relational equation for both the line and plane.

e) Let $\tilde{\mathbf{a}} = \frac{\vec{\mathbf{b}} \times \vec{\mathbf{c}}}{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \times \vec{\mathbf{c}}}$, $\tilde{\mathbf{b}} = \frac{\vec{\mathbf{c}} \times \vec{\mathbf{a}}}{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \times \vec{\mathbf{c}}}$, and $\tilde{\mathbf{c}} = \frac{\vec{\mathbf{a}} \times \vec{\mathbf{b}}}{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \times \vec{\mathbf{c}}}$, where the arrows on $\mathbf{a}, \mathbf{b}, \mathbf{c}$ distinguish $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}$ from their *covectors* $\tilde{\mathbf{a}}, \tilde{\mathbf{b}}, \tilde{\mathbf{c}}$. This definition of $\tilde{\mathbf{a}}$ is consistent with above if $\mathbf{A} = \vec{\mathbf{b}} \times \vec{\mathbf{c}}$. Calculate the nine combinations of $(\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}})^T \cdot (\tilde{\mathbf{a}}, \tilde{\mathbf{b}}, \tilde{\mathbf{c}}) = I$, i.e. $\vec{\mathbf{a}} \cdot \tilde{\mathbf{a}} = 1, \vec{\mathbf{a}} \cdot \tilde{\mathbf{b}} = 0, \dots$ to show they are *mutually orthogonal*. In this sense, $(\tilde{\mathbf{a}}, \tilde{\mathbf{b}}, \tilde{\mathbf{c}})$ is the *dual* or *reciprocal* basis of $(\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}})$.

f) The *contravariant components* of \mathbf{x} are defined as the components (α, β, γ) that satisfy the equation $\mathbf{x} = \vec{\mathbf{a}}\alpha + \vec{\mathbf{b}}\beta + \vec{\mathbf{c}}\gamma$; i.e., $(\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}})$ is the contravariant basis. Using $\mathbf{x} \cdot \tilde{\mathbf{a}}$, etc., calculate the three contravariant components of \mathbf{x} in terms of dot products. Likewise, find the *covariant components* $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ of \mathbf{x} , defined as the components $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ which satisfy the equation $\mathbf{x} = \tilde{\mathbf{a}}\tilde{\alpha} + \tilde{\mathbf{b}}\tilde{\beta} + \tilde{\mathbf{c}}\tilde{\gamma}$; i.e. $(\tilde{\mathbf{a}}, \tilde{\mathbf{b}}, \tilde{\mathbf{c}})$ is the covariant basis.

2. Rotations. In addition to projection, the cross product *generates rotation*. For example, $\boldsymbol{\omega} \times$ transforms to a rotating reference frame and $\mathbf{r} \times$ appears in torque and angular momentum. Rotations are ubiquitous in all of physics, particularly mechanics. We will explore parallels between the cross product, the unit imaginary i , and matrix generator M in the context of rotations.

a) The *complex plane* $\{\mathbf{w} = (x, y)\}$ is the vector space of *real* and *imaginary* components of complex numbers $w = x + iy \in \mathbb{C}$, where $i^2 = -1$. Identify the two natural *basis vectors* (complex numbers) in this space. In addition to linear structure, show that complex multiplication has the form $w = w_1 w_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2) = x + iy$, matching matrix products of $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$.

b) Show that the dot and cross product of two points $\mathbf{w}_1 = (x_1, y_1)$ and $\mathbf{w}_2 = (x_2, y_2)$ are given by the real and imaginary parts of the complex product $w_1^* w_2 = \mathbf{w}_1 \cdot \mathbf{w}_2 + i(\mathbf{w}_1 \times \mathbf{w}_2)_z$, where

$w^* = x - iy$ is the *complex conjugate* of w . Identify the symmetric and antisymmetric terms of this product. Thus the complex product $|w|^2 = w^*w$ equals the vector square $\mathbf{w} \cdot \mathbf{w}$.

c) Show graphically that the operator $w \rightarrow iw$ rotates the point w 90° CCW about the origin, and that the operator $1 + i d\phi : w \mapsto w + iw d\phi$ preserves the magnitude of w (assuming $d\phi^2 = 0$), but rotates it CCW by the infinitesimal angle $d\phi$.

d) Obtain a finite rotation from an infinite number of $d\phi$ rotations as follows: formally integrate the equation $dw = iw d\phi$ with the initial condition $w|_{\phi=0} = w_0$ to obtain the rotation formula $w(\phi) = R_\phi w_0$, where $R_\phi = e^{i\phi}$. Use this result to justify the identity $e^x = \lim_{n \rightarrow \infty} (1 + \frac{x}{n})^n$. Show this is a rotation by separating the Taylor expansion of $e^{i\phi}$ into $x + iy$ to prove Euler's formula, $e^{i\phi} = \cos \phi + i \sin \phi$.

e) Show that complex multiplication by i is equivalent to the vector operator $\hat{\mathbf{z}} \times$, and determine the matrix representation M_z of this operator; ie. $M_z \mathbf{r} = \hat{\mathbf{z}} \times \mathbf{r}$. Do the same for M_x and M_y to show that $\mathbf{v} \times = \mathbf{v} \cdot \mathbf{M} = v_x M_x + v_y M_y + v_z M_z$ is the matrix representation of $\mathbf{v} \times$ for any vector \mathbf{v} . Answer: $\mathbf{M} \sim (M_i)_{jk} = \varepsilon_{ijk}$, the Levi-Civita (cross product), tensor which is completely antisymmetric in indices i, j, k .

f) Restricted to the xy -plane, show that $M_z^2 = -I$, a 2×2 matrix, analogous to $i^2 = -1$, and thus the matrix for a CCW rotation ϕ is $R_\phi = e^{M_z \phi} = I \cos \phi + M_z \sin \phi = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}$. Note that the *exponential* of a matrix M_z is defined by its Taylor expansion. The *Rodrigues formula* for the matrix $R_{\mathbf{v}}$ of a CCW rotation by an angle $v = |\mathbf{v}|$ about the $\hat{\mathbf{v}}$ -axis is $R_{\mathbf{v}} = e^{\mathbf{M} \cdot \mathbf{v}} = I \cos v + \mathbf{M} \cdot \hat{\mathbf{v}} \sin v + \hat{\mathbf{v}} \hat{\mathbf{v}}^T (1 - \cos v)$. The third term projects out the axis of rotation $\hat{\mathbf{v}}$.

3. Stretches. In analogy with the *polar decomposition* $w = x + iy = \rho e^{i\phi}$ of complex numbers, any matrix A can be decomposed $A = RS$ into a stretch S and a rotation R , which are the building blocks of all linear operators.

a) Use the exponential form $R_{\mathbf{v}} = e^{\mathbf{M} \cdot \mathbf{v}}$ and the asymmetry of the generator $M^\dagger = -M$ to show that $R^\dagger R = I$ or $R^\dagger = R^{-1}$. In contrast, a stretch is symmetric or *Hermitian* $S^\dagger = S$.

b) A symmetric matrix S is guaranteed to have a complete set of eigenvectors $V = (\mathbf{v}_1, \mathbf{v}_2, \dots)$ and corresponding eigenvalues $\lambda_1, \lambda_2, \dots$, such that $S\mathbf{v}_i = \mathbf{v}_i \lambda_i$. Show that we can extend this equation to $SV = VW$, where $W = \text{diag}(\lambda_1, \lambda_2, \dots)$ is *diagonal*.

c) Show that two eigenvectors $\mathbf{v}_i, \mathbf{v}_j$ of S with distinct eigenvalues $\lambda_i \neq \lambda_j$ must be orthogonal: $\mathbf{v}_i \cdot \mathbf{v}_j = 0$, and therefore the matrix of eigenvectors is an orthogonal transformation: $V^\dagger V = I$. [bonus]: Interpret the resulting decompositions $S = VWV^\dagger$ and $W = V^\dagger SV$ in terms of rotations.

d) Calculate the eigenvalues and eigenvectors of $M_z = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Do they look familiar? Show that $M_z = VWV^\dagger = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$ and $e^{M_z \phi} = V e^{W\phi} V^\dagger = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$. Multiply this out to verify part 2f). Thus real Hermitian matrices have real eigenvalues while antiHermitian matrices have $\text{Tr}=0$ and imaginary eigenvalues. The exponential of a Hermitian matrix is positive definite with real positive eigenvalues, while the exponential of an antiHermitian matrix is unitary with $\text{Det}=1$ and unit modulus eigenvalues. This is the normal matrix analogy which completes the connection between matrices and complex numbers.

e) *Singular Value Decomposition (SVD)* [bonus] Combine polar ($A = RS$) and eigen ($S = VWV^\dagger$) decompositions, using $U = RV$, to show that any $m \times n$ matrix ${}_m A_n$ may be decomposed into $A = UWV^\dagger$, where $U \equiv V^\dagger R$, $U^\dagger U = I_m$, $V^\dagger V = I_n$, and ${}_m W_n$ is diagonal and *positive definite*.