

University of Kentucky, Physics 404G
Homework #7, Rev. A, due Tuesday, 2021-11-02

1. Gravity waves propagate along the interface between a liquid and a gas (or any two fluids of different density, for example air and water), as gravity or *buoyancy* tries to restore equilibrium. The particles in the liquid follow elliptical trajectories with their amplitude decaying exponentially in depth. If the velocity field $\mathbf{v}(x, z)$ is irrotational ($\nabla \times \mathbf{v} = 0$), it can be represented by the gradient of a scalar flow potential $\mathbf{v} = -\nabla\phi(x, z)$. If the fluid is also incompressible ($\nabla \cdot \mathbf{v} = 0$), its flow satisfies the Laplace equation $\nabla^2\phi = 0$. Let the gas-liquid interface be at height $z = \eta(x, t)$ above the equilibrium level at $z = 0$, and h be water depth (the sea bed is at $z = -h$).

In addition to gravity, surface tension γ , exerts a downward pressure $P = -\gamma\nabla_{\perp}^2\eta$ on the liquid, also propagating ripples on the surface. This is the two-dimensional analog of the vertical force $dF = T f'' dx$ of a wave on an element of string under horizontal tension T . Both of these effects can be described by **Airy wave theory**, which we develop below.

a) Show that the function $\phi(x, z, t) = A \cosh(k(z + h)) \cos(kx - \omega t)$ is a solution of $\nabla^2\phi = 0$. Plot the equipotentials of ϕ at $t = 0$, with arrows showing the direction of \mathbf{v} .

b) Show this solution satisfies the boundary condition $v_z(x, -h) = 0$, at the bottom of the liquid. The boundary condition on the top surface is $v_z = \dot{\eta}$, evaluated at $z = 0$ (the boundary to 0^{th} order). Find A so that ϕ satisfies this boundary condition at the interface $\eta(x, t) = a \sin(kx - \omega t)$.

c) Integrating Newton's law over z leads to Bernoulli's law $\rho\partial_t\phi = \rho g\eta - \gamma\partial_x^2\eta$, where ρ is the mass density of the liquid. Substitute ϕ and η into Bernoulli's law at $z = 0$ to obtain the dispersion relation $\omega^2 = (gk + \frac{\gamma}{\rho}k^3) \tanh(kh)$. Plot $\omega(k)$, $v_\phi(k) = \omega/k$, and $v_g(k) = d\omega/dk$.

d) Calculate the wavelength λ_c below which waves are dominated by surface tension, using $\gamma = 72.8$ mN/m and $\rho = 1.00$ g/cm³ for water. What is the dispersion relation in this limit?

e) Approximate $\phi(x, z, t)$ and $\omega(k)$ in the deep water limit, where $kh \gg 1$. [bonus: Do individual crests move forward or backward within the wave packet?]

f) Approximate $\omega(k)$ in the shallow water limit, and show that all frequencies have the same velocity. What is the speed of a tsunami ($\lambda \approx 100$ km) in 10 km deep [shallow!] ocean waters? How long will it take one wavelength to pass?

2. Elastic waves in solids, also known as body, bulk, **seismic**, stress, or strain waves, are the three-dimensional analog of waves traveling along a Slinky. They have three polarizations: one longitudinal polarized acoustic P-wave, (primary or pressure), and two transverse polarized S-waves (secondary or shear). [Note there are three completely different and inconsistent definitions of S- and P-waves for seismic, optical, and quantum mechanical angular momentum waves!]

Pressure and shear waves are both described in terms of elastic deformation. Similar to the displacement $f(x, t)$ of coils in a Slinky, the displacement field $\mathbf{u}(\mathbf{r}, t)$ describes the shift of the particle at position \mathbf{r} in equilibrium to the new position $\mathbf{r} + \mathbf{u}$. Elastic strain (deformation) ϵ indicates the change in displacement $d\mathbf{u} = \epsilon \cdot d\mathbf{r}$ between neighboring particles, analogous to $f'(x, t)$ for the Slinky. This symmetric *strain tensor* $\epsilon_{ij} \equiv \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ is the matrix of all nine possible linear deformations. On the other hand, the *stress tensor* $\tau_{ij} = dF_i/da_j$ describes all three components of force dF_i per area along all three independent directions of surface area da_j of the interface between two neighboring elements at \mathbf{r} and $\mathbf{r} + d\mathbf{r}$ at equilibrium in the bulk, so that the equal and

opposite force between these elements is $d\mathbf{F} = \pm \boldsymbol{\tau} \cdot d\mathbf{a}$. The normal component of stress to the interface ($i = j$) is called *pressure*, while the tangential components ($i \neq j$) are called *shear*, each propagating their own polarization. Stress transfers energy and momentum across the interface, propagating the wave according to Newton's second law $d\mathbf{F} = dm \mathbf{A}$, which is written $\partial_j \tau_{ij} = \rho \ddot{u}_i$, analogous to $dF = \mu dx \ddot{f}$ for the Slinky.

The generalization of Hooke's law, $dF = \kappa f'' dx$, from the Slinky to the bulk of an elastic material, $\tau_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}$, relates the stress (force) $\boldsymbol{\tau}$ to the strain (stretch) $\boldsymbol{\epsilon}$. This is the most general isotropic and rotationally invariant relation, which contains two independent "spring constant" *Lamé parameters*: λ (pressure), and μ (shear). Other combinations of these constants, each with their own physical interpretation are: Poisson's ratio $\nu = \lambda/2(\lambda + \mu)$, the P-wave (longitudinal, pressure) modulus $M = \lambda + 2\mu$, the S-wave (transverse, shear) modulus $G = \mu$, the *bulk modulus* (compressibility) $K = M - \frac{4}{3}G$, and *Young's modulus* (stretchiness) $E = 2G(1 + \nu) = 3K(1 - 2\nu)$. Nonviscous fluids do not support shear strain ($\mu = 0$) and only accommodate P-waves. For an ideal gas, the isentropic bulk modulus $K_S = \gamma P$ is used, where the *adiabatic index* $\gamma = C_P/C_V$ is the ratio of heat capacities at constant pressure and volume, respectively, and P is the equilibrium pressure. In air at STP, $\gamma = 1.41$ and $P = 1 \text{ bar} = 100 \text{ kPa}$.

a) Combine Hooke's and Newton's laws in the form given above to obtain the wave equation $\rho \ddot{\mathbf{u}} = M \nabla \nabla \cdot \mathbf{u} - G \nabla \times \nabla \times \mathbf{u}$. Note that the two operators $\nabla_{\parallel}^2 \equiv \nabla \nabla \cdot$ and $\nabla_{\perp}^2 \equiv -\nabla \times \nabla \times$ are the longitudinal and transverse projections of the *Laplacian* $\nabla^2 \equiv \nabla \cdot \nabla = \nabla_{\parallel}^2 + \nabla_{\perp}^2$ (the fundamental second vector derivative) that describe the curvature of longitudinal and transverse waves, respectively.

b) For a P-wave, $\nabla \times \mathbf{u} = 0$. Take the divergence of both sides of the wave equation to show that $\rho \ddot{n} = M \nabla^2 n$, where $n \equiv \nabla \cdot \mathbf{u}$ is the compression of the medium. Show that the wave $n = a e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}$ is a solution with amplitude a traveling in the direction $\hat{\mathbf{k}}$, and obtain the dispersion relation $\omega(\mathbf{k})$. Calculate the velocity of a seismic P-wave ($K = 200 \text{ GPa}$, $\rho = 1700 \text{ kg/m}^3$ for rock), and of sound in air ($\rho = 1.225 \text{ kg/m}^3$), water ($K = 2.2 \text{ GPa}$), and steel ($K = 160 \text{ GPa}$, $\rho = 8050 \text{ kg/m}^3$).

c) For an S-wave, $\nabla \cdot \mathbf{u} = 0$. Take the curl of both sides of the wave equation to show that $\rho \ddot{\mathbf{m}} = G \nabla^2 \mathbf{m}$, where $\mathbf{m} \equiv \nabla \times \mathbf{u}$ is the shear strain which is perpendicular to the direction of propagation. Show that the wave $\mathbf{m} = \mathbf{a} e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}$ is a solution with transverse amplitude \mathbf{a} traveling perpendicular to \mathbf{k} , and obtain the dispersion relation $\omega(\mathbf{k})$. Calculate the velocity of a seismic S-wave ($M = 100 \text{ GPa}$ for rock), and of a shear wave in steel ($M = 79.3 \text{ GPa}$). There are no S-waves in water, air, or in the molten *outer core* of the earth, which is how we know it exists.