University of Kentucky, Physics 404G
Homework #8, Rev. A, due Tuesday, 2021-11-16

-1. Newton's Express [bonus] A vivid display of conservation of both energy and momentum is illustrated in Newton's Cradle. For balls of equal mass $m_i = m$, the unique solution conserving both $p = \sum_i m_i v_i$ and $E = \sum_i \frac{1}{2} m_i v_i^2$ is the final ball continuing at the same velocity v_n as the initial velocity v_1 of the first ball. In the 2-ball case with different masses m_1 and m_2, solve for the velocity v_2 of the second ball, initially at rest, after collision with the first ball of velocity v_1. Interpret your results in terms of the *impedance* $Z_i = i\omega m_i$ of each ball, in the limit $\omega \to 0$ of non-periodic motion, noting that $F = Zv$. How does this simplify the analysis of n balls of different mass m_i? Compare your result with the reflected energy of a wave due the mismatch in impedance at the interface of two media.

0. Inertial Express Freeway engineers desire to protect oncoming traffic from a roadside bridge pylon by placing a series of Fitch barriers of sand-filled barrels of increasing mass in front so that a $m_0 = 1000$ kg car on a collision course with the hazard would decelerate uniformly from $v_i = 100$ km/h to $v_f = 10$ km/h, and thus spread the impact out to a constant force over $d = 10$ m.

a) What distribution of mass $\lambda(x) = dm/dx$ should be used, assuming inelastic collisions in the limit of infinitessimally thin barrels, so that the mass distribution is continuous? Assume that the barrels end up at the same speed as the car, but are then ejected so that the the mass of the car remains constant.

b) What is the deceleration profile $a(t)$ for a car of mass $m = 2000$ kg and the same $\lambda(x)$?

c) [bonus: repeat part a) with the car accumulating the mass of each barrel it hits.]

1. Orient Express An imaginary nation practices an esoteric method of interrogation by tying the suspect to a swivel chair, placed a distance a away from a straight infinitely long train track. The angle $\phi(t)$ of the chair is turned to always face an oncoming train, approaching at constant velocity v along the tracks. The suspect has blinders and can only see the train, which appears directly ahead at a distance $\rho(t)$ away. This interrogation method only works as the train approaches from afar, because as it reaches the point of closest approach, it appears to the suspect that the train is mysteriously repelled away by an imaginary force, spoiling any pretense of suspence. Your job as the suspect is to determine from $\rho(t)$ whether or not the train poses an imminent threat, and whether to tell them everything you know or to sit and swivel firm.

a) Use the Lagrangian for a free particle to obtain the equations of motion for $\rho(t)$ and $\phi(t)$.

b) Use the first integrals from conservation of angular momentum $p_\phi = \ell = mr^2\dot{\phi} = mva$ and energy $\mathcal{H} = \epsilon = \frac{1}{2}mv^2$ to reduce the equations of motion to first order. Substitute $\dot{\phi}$ in the radial equation to obtain a first order differential equation ρ independent of ϕ.

c) Separate variables and integrate to obtain $t(\rho)$. Show that the resulting $\rho(t)$ is a solution of the second order equation of motion for ρ.

d) Substitute $\rho(t)$ into the first order ϕ equation and integrate to obtain $\phi(t)$. Show that for initial conditions $\rho_0 = a,\ \phi_0 = 0$, the solution is $x(t) = \rho\cos\phi = a,\ y(t) = \rho\sin\phi = vt$.

2. Despite what we know about earth's outer core being molten (it doesn't propagate earthquake shear waves), the company **Intraterrestrial Express** developed a shady business model to deliver packages 'across' the globe by dropping them through an evacuated tunnel through the center of the earth from southern Argentina to Mongolia. Not many countries are diametrically opposite each other—about the only other options are New Zealand to Spain or Indonesia to Brazil! Ignoring these minor technicalities, we will calculate how long would it take a package dropped from rest to reach the other side, ignoring friction and rotation of the earth.

a) Show that the universal law of gravitation $\boldsymbol{F} = -GMm\hat{\boldsymbol{r}}/r^2$ satisfies Gauss' law just as the electrostatic force $\boldsymbol{F} = Qq\hat{\boldsymbol{r}}/4\pi\epsilon_0 r^2$ does. Use Gauss' law to show that the force inside the earth is springy: $F(r) = -kr$, and determine the effective spring constant of the earth. Integrate $\boldsymbol{F}(r)$ to determine the potential $V(r) = -\int_0^r \boldsymbol{F}(r) \cdot \boldsymbol{dr}$.

b) Using conservation of energy $\epsilon = \frac{1}{2}mv^2 + V(r)$, integrate $dt = dr/v$ to obtain the ETA Δt of the package. Invert the general expression $t(r)$ to obtain the trajectory $r(t)$.

c) Each galaxy is immersed in a spherical **dark matter halo** of uniform density, which affects the rotation curves of luminous matter in the galaxy. Plot the velocity $v(r)$ of stars in circular orbits as a function of radius from the center of the galaxy for three models: i) a supermassive black hole at the center of the galaxy dominates the mass distribution, [bonus: ii) the mass is dominated by the stars in a disk of uniform surface density], iii) the mass is dominated by the dark matter halo.

3. After recovering from most certain bankruptcy, the newly organized company **Extraterrestrial Express** settled on a more practical, but nonetheless shady, technique to deliver 'packages' in the cargo hold of an ICBM. Follow the same steps as in problem #1, but now with a potential $V(r)$, to calcualte $\rho(t)$ and therefore $\phi(t)$. This gives the formulas for Keplerian motion. In the limit of a near-earth orbit, how long would it take to deliver the same package as in problem #2? [bonus: What combinations of initial angle and velocity should the company avoid at all cost?]

4. [bonus] In addition to angular momentum $\boldsymbol{\ell}$ and energy E, the Laplace-Runge-Lenz vector $\vec{\boldsymbol{A}} = \vec{\boldsymbol{p}} \times \vec{\boldsymbol{\ell}} - m\gamma\hat{\boldsymbol{\rho}}$ is a conserved quantity of Kepler orbits. It is a *dynamic*, not *geometric* conserved quantity because there is no coordinate system in which it is the generalized momentum of a *cyclic* coordinate in the Lagrangian. However by Noether's theorem, it still corresponds to rotational symmetry in a higher 4-dimensional space. This symmetry was used by Pauli to solve for the energies of the hydrogen atom even before the Schrödinger equation was published!

a) Show that the coordinate transformation ... varies the Lagrangian as $L' =$ ....

b) Apply Noether's theorem to obtain the conserved current $\vec{\boldsymbol{A}}$.

c) Using $\dot{\vec{\boldsymbol{p}}} = \vec{\boldsymbol{F}}$ and $\dot{\vec{\boldsymbol{\ell}}} = I\vec{\boldsymbol{\omega}}$, show that $\frac{d}{dt}\vec{\boldsymbol{p}} \times \vec{\boldsymbol{\ell}} = \frac{d}{dt}m\gamma\hat{\boldsymbol{\rho}}$, comfirming that $\vec{\boldsymbol{A}}$ is a constant of the motion.

d) Show that $\vec{\boldsymbol{\ell}} \cdot \vec{\boldsymbol{A}} = 0$ and $A^2 = m^2\gamma^2 + 2mE\ell^2$; thus the 7 constants of motion $(E, \vec{\boldsymbol{\ell}}, \vec{\boldsymbol{A}})$ provide only 5 constraints to the 6 degrees of freedom $(\vec{\boldsymbol{r}}, \vec{\boldsymbol{v}})$. The final degree of freedom (inital time, t_0) does not corespond to a constant of the motion.

e) Derive the elliptical orbit equation $c = \rho(1 + e\cos\phi)$ by evaluating $\vec{\boldsymbol{\rho}} \cdot \vec{\boldsymbol{A}}$, and finding c and e in terms of E, ℓ, and A.

f) Since the $\hat{\boldsymbol{\rho}}$ lies on a unit circle, the momentum vectors $\vec{\boldsymbol{p}}(\phi)$ also lie on a circle with radius $m\gamma/\ell$, offset by A/ℓ from the origin (see Fig. 1). Derive the equation for the circle by isolating the $\hat{\boldsymbol{\rho}}$ term from $\vec{\boldsymbol{A}}$ and squaring both sides of the equation. This symmetry in $\vec{\boldsymbol{p}}$ is related to the above Noether symmetry.