University of Kentucky, Physics 404G
Homework #10, Rev. A, due Sunday, 2021-12-05

1. Rotating room—Experience first-the inertial forces in a rotating reference frame!

a) Compare the radius of curvature of a ball rolling on the table with calculations.

b) Calculate the centrifugal acceleration as a function of radius by measuring the angle of a pendulum hanging from the ceiling, and compare with calculations.

2. Nuclear Magnetic Resonance (NMR) is a high-precision spectroscopy technique developed in 1946 by Edward Purcell and Felix Bloch. This technology is used in Magnetic Resonance Imaging (MRI) machines. In NMR, the spin $\boldsymbol{s}$ of an atomic nucleus precesses about a constant magnetic field B_0 similar the precession of a spinning top. This characteristic resonance called the *Larmor frequency* $\omega_L = \gamma B$ is proportional to B_0 and to the gyromagnetic ratio γ of the magnetic moment $\boldsymbol{\mu}$ to the angular momentum $\boldsymbol{s}$ of the nucleus, ie. $\boldsymbol{\mu} = \gamma \boldsymbol{s}$.

a) Show that $\gamma = e/2m$ for a charged point particle in a circular orbit, independent of radius r. This does not hold for a composite particle; we define its *g-factor* by $\gamma = g \cdot e/2m$, ie. how much larger than for a point particle of the same mass and charge. A pointlike quantum mechanical particle with spin $s = \hbar/2$ has $g = 2$. For example, the electron has $g_e = 2.002319$, which is slightly larger than 2 due to vacuum polarization. The neutron also has spin $s = \hbar/2$, but $g_n = -3.826$ due to its internal quark structure. Thus its magnetic moment is $\mu_n = g(e/2m)(\hbar/2) = (g/2)\mu_N = -1.91\,\mu_N$. The unit of magnetic moment used in nuclear physics is the *nuclear magneton* $\mu_N = e\hbar/2m_p$, which equals the magnetic moment of a pointlike proton with orbital angular momentum $\ell = \hbar$ (p-orbital).

b) Solve the classical equation of motion (the Bloch equation) $\dot{\boldsymbol{s}} = \boldsymbol{\tau} = \boldsymbol{\mu}_n \times \boldsymbol{B} = \gamma_n \boldsymbol{s} \times \boldsymbol{B}$ for the *Larmor precession* of a neutron in a constant magnetic field $\hat{\boldsymbol{z}} B_0$, with initial spin $\boldsymbol{s}_0$ at $t = 0$. Compare this with the precession of a spinning top.

c) The mechanical equations of motion are simplified in a rotating reference frame, where $(\,\hat{\boldsymbol{x}}' \quad \hat{\boldsymbol{y}}'\,) = (\,\hat{\boldsymbol{x}} \quad \hat{\boldsymbol{y}}\,)\begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix}$. [Rabi, Ramsey, Schwinger, Rev. Mod. Phy. **26**, 167, 1954]. Since the operator $\boldsymbol{\omega} dt \times$ generates this rotation, the time derivative becomes $\dot{\boldsymbol{s}} = \dot{\boldsymbol{s}}' + \boldsymbol{\omega} \times \boldsymbol{s}$, where $\dot{\boldsymbol{s}}'$ is of components in the rotating frame. Substitute $\dot{\boldsymbol{s}}$ into the Bloch equation and show that the form remains the same except for the replacement of $\boldsymbol{B}$ with the effective field $\boldsymbol{B}' = \boldsymbol{B} + \boldsymbol{\omega}/\gamma_n$. Note this field is zero if the frame is rotating at the Larmor frequency $\boldsymbol{\omega}_L = -\gamma_n \boldsymbol{B}$, and thus the spin remains constant $\boldsymbol{s}' = \boldsymbol{s}_0$. Reconcile this picture with the solution in the static frame.

d) The z-component of spin s_z does not change in a constant magnetic field $\hat{\boldsymbol{z}} B_0$ (called a *holding field* because it preserves the spin state). To transition the spin from up to down, we must use an oscillatory (RF) field $B_1(\hat{\boldsymbol{x}} \cos\omega t + \hat{\boldsymbol{y}} \sin\omega t)$. In the rotating frame with angular velocity $\hat{\boldsymbol{z}}\omega$, show that the total field is $\boldsymbol{B}' = \hat{\boldsymbol{z}}'(B_0 + \omega/\gamma_n) + \hat{\boldsymbol{x}}' B_1$, which is constant. Let θ be the angle between $\boldsymbol{B}'$ and $\hat{\boldsymbol{z}}$, and let the initial spin be $\boldsymbol{s}_0 = \hat{\boldsymbol{z}}$. Show that the z-component of the spin varies as $s_z(t) = s_0(\cos^2\theta + \sin^2\theta \cos\gamma_n B' t) = s_0(1 - 2\sin^2\theta \sin^2(\gamma_n B' t/2))$, which oscillates at the *Rabi flopping frequency* $\omega_R = \gamma_n B'$. Plot the amplitude of oscillation as a function of the RF frequency ω and note the resonance at $\omega = \omega_L$. [bonus: Plot the 3-d trajectory of $\boldsymbol{s}$ in the lab frame over half a Rabi cycle.]