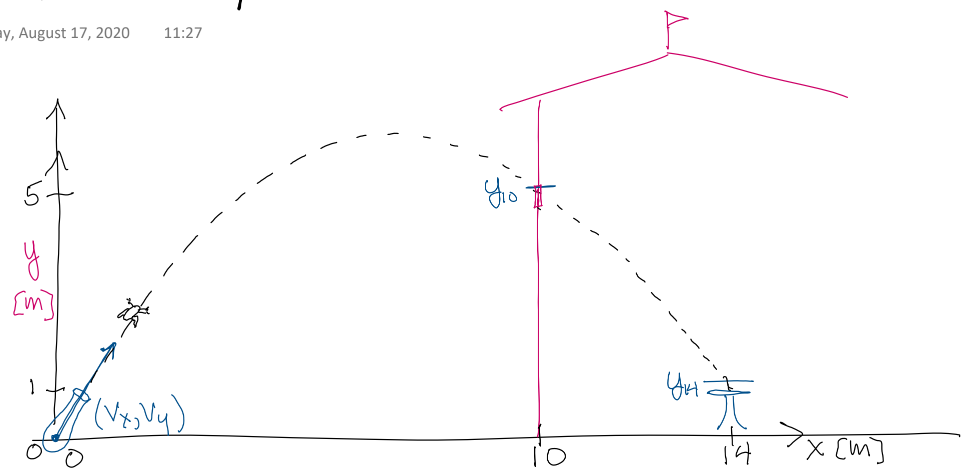
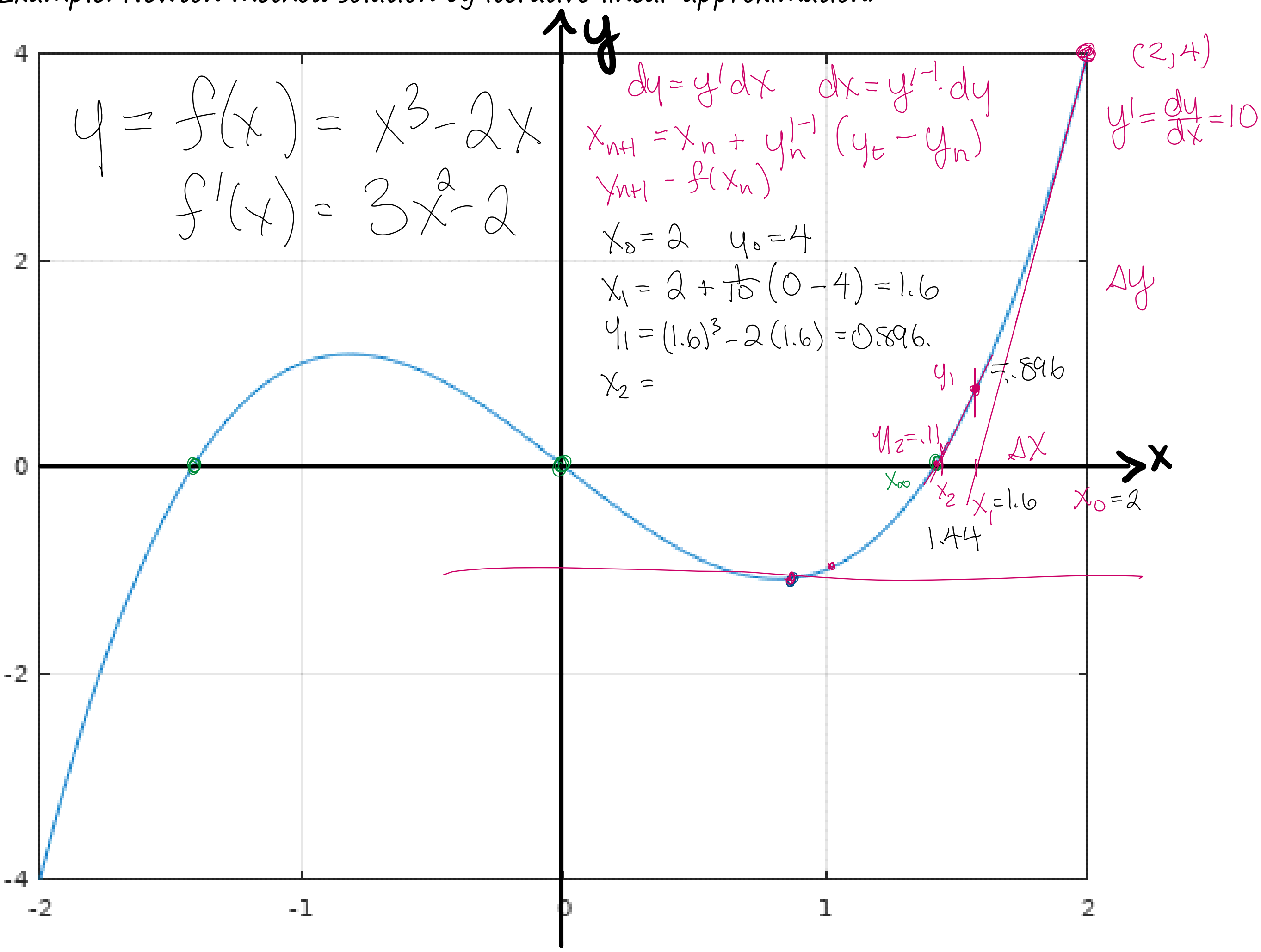
L03 Newton-Raphson

Monday, August 17, 2020 11:27



Example: Newton method solution by iterative linear approximation.



Not everything is linear, BUT all differentials are linear by definition.

Newton-Raphson method does the exact same thing with vectors instead of scalars.
The relevant derivative is the Jacobian matrix of all partial derivatives.

$$y_{10} = -60 + 13 v_x + 2 v_y - v_x^2$$
$$y_{14} = -272 + 7 v_x + 31 v_y - v_y^2$$

$$\begin{bmatrix} y_{10} \\ y_{14} \end{bmatrix} = f\left(\begin{bmatrix} v_x \\ v_y \end{bmatrix} \right)$$

$$y = f(x) \qquad\qquad \vec{y} = f(\vec{v})$$
$$dy = f'(x)\, dx \qquad \Longrightarrow \qquad d\vec{y} = J\, d\vec{v}$$

$$J = \begin{array}{c} \\ \partial y_{10} \\ \partial y_{14} \end{array} \begin{array}{c} \begin{array}{cc} /\partial v_x & /\partial v_y \end{array} \\ \begin{pmatrix} \dfrac{\partial y_{10}}{\partial v_x} & \dfrac{\partial y_{10}}{\partial v_y} \\ \dfrac{\partial y_{14}}{\partial v_x} & \dfrac{\partial y_{14}}{\partial v_y} \end{pmatrix} \end{array}$$

$$\begin{pmatrix} \Delta y_{10} \\ \Delta y_{14} \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 7 & 3 \end{pmatrix} \begin{pmatrix} \Delta v_x \\ \Delta v_y \end{pmatrix} \qquad \frac{\partial y_{10}}{\partial v_x} = 13 - 2 v_x = 3$$

$$@\ \vec{v} = (5, 14)$$

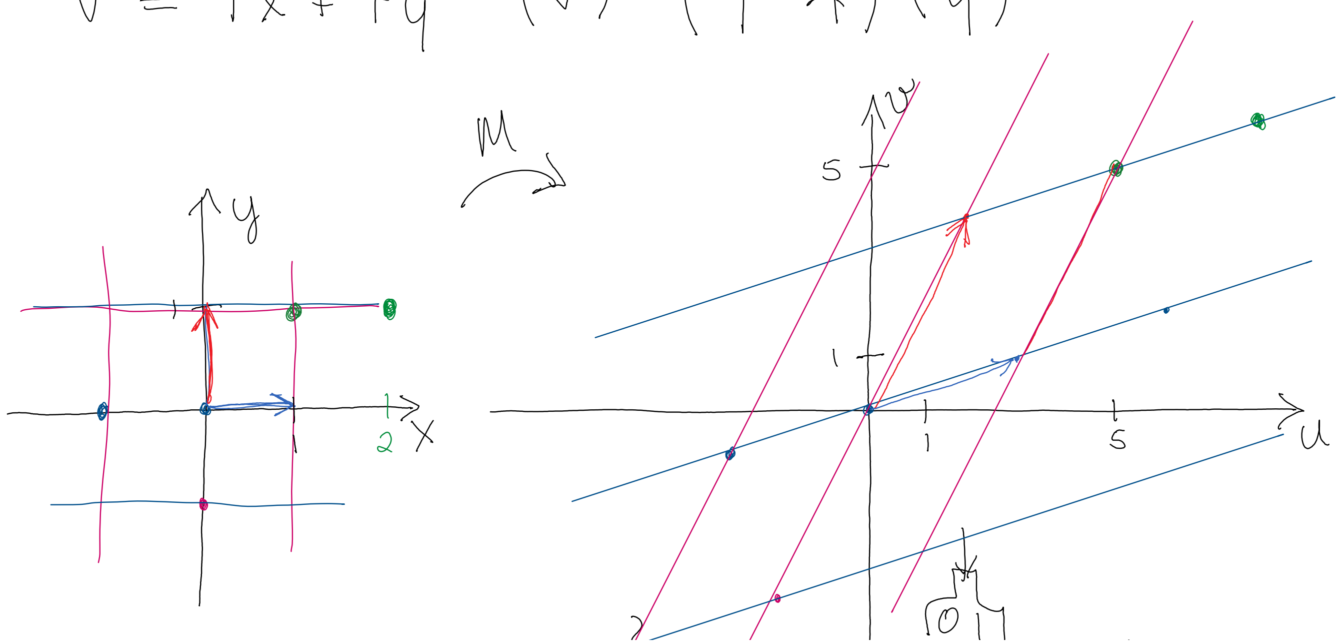
$$\Delta \vec{v} = J^{-1} \Delta \vec{y} \qquad \vec{v}_{n+1} = \vec{v}_n + J^{-1} \cdot (\vec{y}_t - \vec{y}_n) \qquad \vec{y}_{n+1} = f(\vec{v}_{n+1})$$

$$f'(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon) - f(x)}{x + \varepsilon - x} = \frac{f(x+\varepsilon) - f(x)}{\varepsilon}$$

Matrices as operators and matrix multiplication.

$$(u, v) = f(x, y) \qquad \vec{w} = M \quad \vec{z}$$

$$u = 3x + 2y$$
$$v = 1x + 4y \qquad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$



$$\begin{pmatrix}5\\5\end{pmatrix}=\begin{pmatrix}3&2\\1&4\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\qquad \begin{pmatrix}8\\6\end{pmatrix}=\begin{pmatrix}3&2\\1&4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}\qquad \begin{pmatrix}3&2\\1&4\end{pmatrix}=\begin{pmatrix}3&2\\1&4\end{pmatrix}\begin{pmatrix}1&\boxed{0}\\0&\boxed{1}\end{pmatrix}$$

$$\begin{pmatrix}8\\6\end{pmatrix}=\begin{pmatrix}3&2\\1&4\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}$$

$$\begin{pmatrix}q\\r\end{pmatrix}=\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}\qquad \begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}3&2\\1&4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}\qquad \begin{pmatrix}q\\r\end{pmatrix}=\underbrace{\overset{Q}{\begin{pmatrix}1&2\\3&4\end{pmatrix}}\overset{M}{\begin{pmatrix}3&2\\1&4\end{pmatrix}}}\begin{pmatrix}x\\y\end{pmatrix}$$

$$\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}$$

$$QM \longrightarrow \begin{pmatrix}5&10\\13&22\end{pmatrix}$$

$$\begin{pmatrix}x\\y\end{pmatrix}=L\begin{pmatrix}u\\v\end{pmatrix}\qquad \begin{pmatrix}u\\v\end{pmatrix}=M\begin{pmatrix}x\\y\end{pmatrix}\qquad \begin{pmatrix}x\\y\end{pmatrix}=LM\begin{pmatrix}x\\y\end{pmatrix}\qquad LM=I$$

$$M^{-1}M=I$$

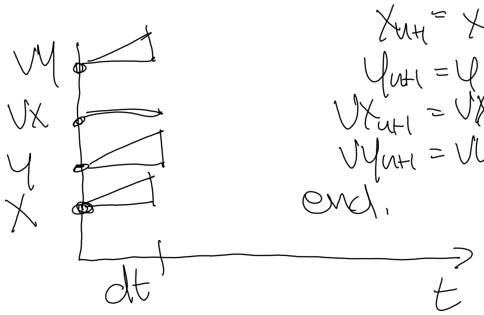
$$M=\begin{pmatrix}3&2\\1&4\end{pmatrix}\qquad \det M = 3\cdot 4-2\cdot 1 = 10\qquad M^{-1}\quad \frac{1}{10}\begin{pmatrix}4&-2\\-1&3\end{pmatrix}$$

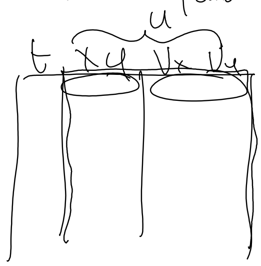
$$M^{-1}\cdot M=\frac{1}{10}\begin{pmatrix}4&-2\\-1&3\end{pmatrix}_2\cdot\begin{pmatrix}3&2\\1&4\end{pmatrix}_2=\frac{1}{10}\begin{pmatrix}10&0\\0&10\end{pmatrix}_2=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$\begin{pmatrix}x\\y\end{pmatrix}=M^{-1}\begin{pmatrix}u\\v\end{pmatrix}=\frac{1}{10}\begin{pmatrix}4&-2\\-1&3\end{pmatrix}\begin{pmatrix}8\\6\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix}$$

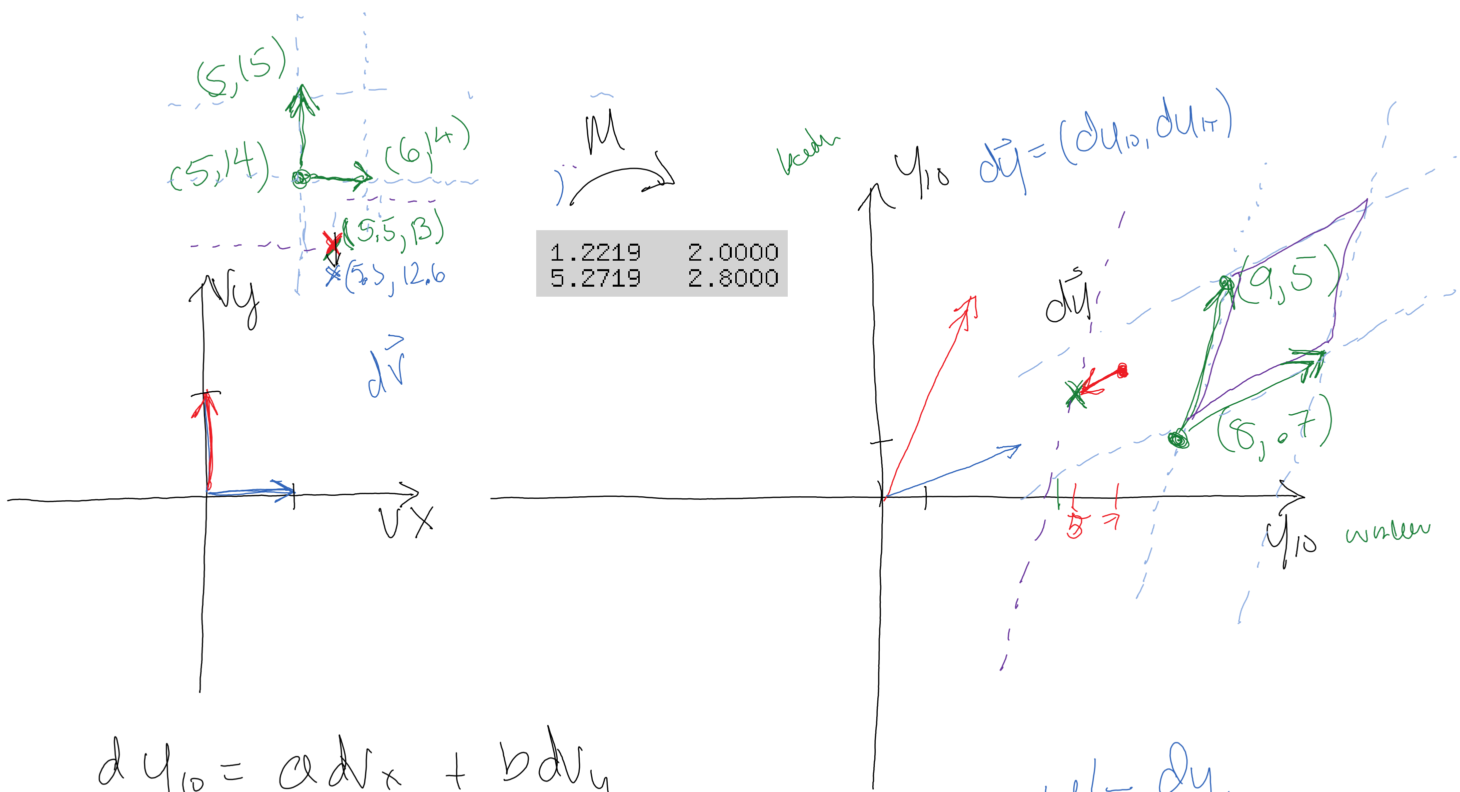
Application to the frog-prince-cannon shooting problem:

while ($x_{end} < 14$)
$t_{n+1} = t_n + dt$
$x_{n+1} = x_n + dt \times vx_n$
$y_{n+1} = y_n + dt \times vy_n$
$vx_{n+1} = vx_n + dt \times ax_{const}$
$vy_{n+1} = vy_n + dt \times ay_{const}$
end.





$$u'(t,u) = f(t,u)$$



1.2219	2.0000
5.2719	2.8000

$$dy_{10} = a\,dv_x + b\,dv_y$$

$$dy_{14} = c\,dv_x + d\,dv_y$$

$$d\begin{pmatrix} y_{10} \\ y_{14} \end{pmatrix} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{J} d\begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

$$d\vec{y} = J\,d\vec{v}$$

$$d\vec{v} = J^{-1}\,d\vec{y}$$

$$y' = \frac{dy}{dx}$$

$$dy = y' \cdot dx$$

$$J = \begin{pmatrix} \frac{d\vec{y}}{dv_x} & \frac{d\vec{y}}{dv_y} \end{pmatrix}$$