L06 Index Notation

Wednesday, August 26, 2020 15:26

* Matrix multiplication $\mathbb{W} = \mathbb{M}\,\mathbb{Z}$

$$u = a x + b y \qquad \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$v = c x + d y$$

$$w_1 = M_{11} z_1 + M_{12} z_2 \qquad \underset{2\;\;\;1}{\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}} = \underset{2\;\;\;2}{\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}} \underset{2\;\;\;1}{\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}}$$

$$w_2 = M_{21} z_1 + M_{22} z_2$$

$$w_1 = \sum_{j=1}^{2} M_{1j} z_j$$

$$w_2 = \sum_{j=1}^{2} M_{2j} z_j$$

M(i,j) - Fortran, Matlab
M[i,j] - Python, Julia
M[i][j] - C/C++, Java
M[[i,j]] - Mathematica

$$w_i = \sum_{j=1}^{2} M_{ij} z_j$$

$$= M_{ij} z_j$$

$$\mathbb{W} = \mathbb{M}\,\mathbb{Z}$$

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for i=1:3
  w(i)=0;
  for j=1:3
    w(i)=w(i)+M(i,j)*z(j);
  end
end
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$$w^i = M^i{}_j z^j \qquad M^{T}{}_i{}^j = M^j{}_i$$

"contravariant and covariant indices"

* Inner product

$$\vec{v}\cdot\vec{w} = v_1 w_1 + v_2 w_2 + \dots = v_i w^i = \begin{pmatrix} v^1 & v^2 \end{pmatrix} \begin{pmatrix} w^1 \\ w^2 \end{pmatrix}$$

$$\text{or} = v^i g_{ij} w^j = \begin{pmatrix} v^1 & v^2 \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} w^1 \\ w^2 \end{pmatrix}$$

* Identity matrix & orthonormality

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbb{I}\ \text{identity}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \mathbb{I}\, z = z \qquad I_{ij} \equiv \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

$$\delta^i{}_j \, z^j = z^i$$

$$\vec{v} \cdot \vec{w} = \overset{\text{orthonormal}}{\underbrace{v^i \hat{e}_i}_{\vec{v}} \cdot \underbrace{\hat{e}_j w^j}_{\vec{w}}} = v^i \delta_{ij} w^j = v^i w_i \qquad \hat{e}_i \cdot \hat{e}_j = \delta_{ij}$$

* Determinants, cross/triple products.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

[2 terms] = 2!

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \begin{matrix} a & b \\ d & e \\ g & h \end{matrix} = aei + bfg + cdh - ceg - afh - bdi$$

[6 terms] = 3!

[24 terms] = 4!

$$\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix} = a \cdot \begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} \begin{matrix} f & g \\ j & k \\ n & o \end{matrix} - b \cdot \begin{vmatrix} e & g & h \\ i & k & l \\ n & o & p \end{vmatrix} + c \begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - d \begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}$$

$\pi_1 = 1 \quad \pi_2 = 3 \quad \pi_3 = 2 \quad \pi_4 = 4$

$\pi : \begin{matrix} 1 & 2 & 3 & 4 \\ \downarrow & \times & & \downarrow \\ 1 & 2 & 3 & 4 \end{matrix} \in S_n$ the group of $n!$ permutations of $(1, 2, \ldots n)$

$$\det A = \sum_{\pi} \text{sign}(\pi)\, A_{1\pi_1} A_{2\pi_2} \cdots A_{n\pi_n}$$

sign $= \pm 1$ for even / odd # of transpositions

$$= \sum_{i_1 i_2 \cdots i_n} \underbrace{\varepsilon_{i_1 i_2 \cdots i_n}}\, A_{1 i_1} A_{2 i_2} \cdots A_{n i_n}$$

Levi-Civita Tensor (totally antisymmetric) $= \begin{cases} 1 & \text{for even permutation} \\ -1 & \text{for odd permutation} \\ 0 & \text{for repeated index} \end{cases}$

$\Rightarrow \varepsilon_{ij} A_{1i} A_{2j}$ for 2×2 matrix, $\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$= \cancel{\varepsilon_{11} A_{11} A_{21}} + \overset{(+)}{\varepsilon_{12} A_{11} A_{22}} + \overset{(-)}{\varepsilon_{21} A_{12} A_{21}} + \cancel{\varepsilon_{22} A_{12} A_{22}}$$

$\Rightarrow \varepsilon_{ijk} A_{1i} A_{2j} A_{3k}$ for 3×3 matrix, ε_{ijk} $3\times 3\times 3$

$$\vec{c} = \vec{a} \times \vec{b} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \varepsilon^{i}{}_{jk}\, \hat{e}_i\, a^j b^k \quad \text{OR} \quad c^i = \varepsilon^{i}{}_{jk}\, a^j b^k$$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \varepsilon_{ijk}\, a^i b^j c^k$$

* vector derivatives

$$\nabla = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \Rightarrow \nabla_i = \frac{\partial}{\partial q^i} \equiv \partial_i \quad \text{or} \quad {}_{,i}$$

$$\nabla f = \hat{e}_i\, \partial_i f \qquad \text{or} \qquad (\nabla f)_i = \partial_i f = f_{,i}$$

$$\nabla \times \vec{A} = \varepsilon^{ij}{}_{k}\, \hat{e}_i\, \partial_j A^k \quad \text{or} \quad (\nabla \times \vec{A})^i = \varepsilon^{ij}{}_{k}\, A^k{}_{,j}$$

$$\nabla \cdot \vec{A} = \partial_i A^i = A^i{}_{,i}$$

$$\nabla^2 f = \nabla \cdot \nabla f = \partial_i \partial^i f = f_{,i}{}^{,i} \quad \text{or} \quad f_{,ii}$$

* identities

0. $\delta_{ij} = \delta_{ji}$

1. $a_{\cdot\cdot i \cdot\cdot}\, \delta_{ij} = a_{\cdot\cdot j \cdot\cdot}$

2. $\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij} = -(\varepsilon_{jik} = \varepsilon_{ikj} = \varepsilon_{kji})$

3. $\varepsilon^{kij}\, \varepsilon_{kmn} = \delta^{ij}_{mn} \equiv \delta^i{}_m \delta^j{}_n - \delta^i{}_n \delta^j{}_m$

* proof of the BAC-CAB rule: (ignoring index height)

$$(\vec{a} \times (\vec{b} \times \vec{c}))_i = \varepsilon_{ijk} a_j (\vec{b} \times \vec{c})_k$$ [first x indices]

$$= \varepsilon_{ijk} a_j \varepsilon_{kmn} b_m c_n$$ [second x indices]

$$= \varepsilon_{kij} \varepsilon_{kmn} a_j b_m c_n$$ [rearranging]

$$= (\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) a_j b_m c_n$$ [identity]

$$= a_n b_i c_n - a_m b_m c_i$$ [contracting δ's]

$$= b_i \vec{a} \cdot \vec{c} - c_i \vec{a} \cdot \vec{b}$$ [interpreting ·]

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b} (\vec{a} \cdot \vec{c}) - \vec{c} (\vec{a} \cdot \vec{b})$$ [as vectors]