L07 Rotations w/Complex Numbers, Matrices

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COMPLEX NUMBERS

- Imaginary numbers

$(x^2-1) = (x+1)(x-1) = 0$ has 2 roots: $x = \pm 1$

$(x^2+1) = \; ? \; = 0$ needs solutions of $x^2 = -1$!

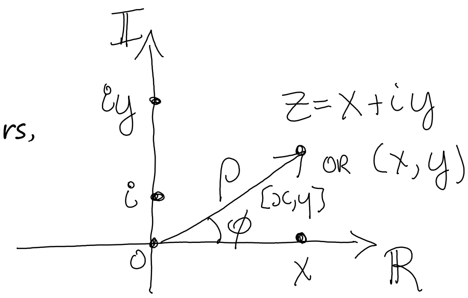
$\nexists\, x \in \mathbb{R} \ni x^2 = -1$ ie $x = \pm\sqrt{-1}$

So define the new imaginary number $i = \sqrt{-1}$ then $x = \pm i$ are the solutions,

and $x^2+1 = (x+i)(x-i)$ now factors into two binomials like x^2-1.

Likewise, $x^2+2=0 \Rightarrow x = \pm\sqrt{2}\, i$ so we have a whole imaginary line

$i\mathbb{R} = \{ yi \mid y \in \mathbb{R} \}$ analogous to the real line $\mathbb{R}$



Real and imaginary numbers and add separately like vectors,

so the direct sum $\mathbb{C} = \{ z = x+iy \mid x,y \in \mathbb{R} \}$

forms a vector space called the "Complex plane" (both real and imaginary).

$\vec{z}_1 = (\hat{x}x_1 + \hat{y}y_1) \qquad z_1 = x_1 + iy_1 \qquad z_1+z_2 = (x_1+x_2) + i(y_1+y_2)$

$\vec{z}_2 = (\hat{x}x_2 + \hat{y}y_2) \qquad z_2 = x_2 + iy_2 \qquad \alpha z_1 = (\alpha x_1) + i(\alpha y_1)$

- Complex algebra:

Complex numbers have the extra structure $i^2 = -1$ beyond standard vector arithmetic.

Thus $z_1 \cdot z_2 = (x_1+iy_1)(x_2+iy_2) = \underbrace{(x_1x_2 - y_1y_2)}_{x} + i\underbrace{(x_1y_2 + x_2y_1)}_{y}$

This is similar to multiplying a vector by a matrix, except that a matrix has 4 degrees of freedom (each element), while the complex number only has two:

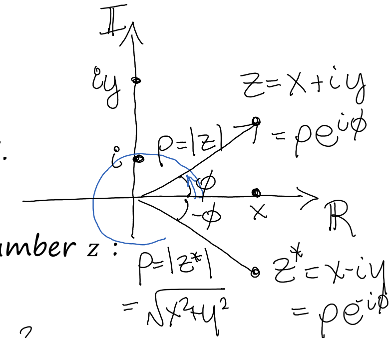
$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} & z_1 & \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

$1 \cdot z$ [identity] $\qquad i \cdot z$ [CCW rotation by 90°]

Thus multiplication by 'i' is similar to the cross product and also generates rotations (H02#2).

Either of these operations can be scaled by multiplying by a real number.
Unlike matrix multiplication, these operations preserve angles and are called "conformal".
Thus we can use complex functions to form various orthogonal coordinate systems.

- Conjugate and Polar coordinates:



The complex conjugate $z^* = (x - iy)$ sends 'i' to '-i'.

It is used find the 'magnitude' or absolute value |z| of a complex number z :

$$|z|^2 \equiv z^* z = (x - iy)(x + iy) = x^2 + y^2$$

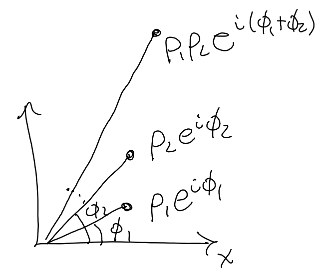
The 'argument' or angle can be found using normal trig:

$$\tan\phi = y/x \qquad x = \rho\cos\phi \qquad y = \rho\sin\phi$$

Euler's identity $e^{i\phi} = \cos\phi + i\sin\phi$ combines these expressions:

$$z = \rho e^{i\phi} = \rho\cos\phi + i\rho\sin\phi = x + iy$$

The product (and thus exponentials) are simpler in polar coordinates
(also double-angle formulae):



$$z_1 \cdot z_2 = (\rho_1 e^{i\phi_1})(\rho_2 e^{i\phi_2}) = (\rho_1\rho_2) e^{i(\phi_1+\phi_2)}$$

Multiplying by $e^{i\phi}$ rotates counterclockwise by the angle ϕ

and multiplying by ρ stretches by the factor ρ.

The complex conjugate in polar coordinates is: $\rho e^{-i\phi} = x - iy$.

- The universality of complex numbers is captured by the Fundamental Theorem of Algebra: Any n-th order polynomial can be factored into a product of n binomials over $\mathbb{C}$: and therefore has n complex roots (accounting for multiplicity). If the coefficients are real then the roots or either real or occur in complex-conjugate pairs.

$$P_n(z) = a_n z^n + a_{n-1} z^{n-1} + \dots a_2 z^2 + a_1 z^1 + a_0$$
$$= a_n (z - z_1)(z - z_2) \cdots (z - z_n)$$

Thus complex numbers solve a much larger problem than the original one posed.

- Quaternions were invented by William Rowan Hamilton (also known for the Hamiltonian) to generalize rotations to 3 dimensions. Seeing that 1 and i represent the $\hat{x}$ and $\hat{y}$, he tried unsuccessfully to add a third imaginary number j to represent $\hat{z}$.
  He later had an epiphany to use a 4-vector $a1 + bi + cj + dk$ with a separate imaginary (i, j, k) for each $(\hat{x}, \hat{y}, \hat{z})$ axis, and carved the famous inscription $i^2 = j^2 = k^2 = ijk = -1$ into the Brougham (Broom) Bridge, Dublin. The numbers are associative: $(ij)k = i(jk)$, but not commutative. For example, multiplying by $-k$ gives $ijk(-k) = ij(-k^2) = ij = k$, but multiplying by ji on the left gives $(ji)ijk = k = -ji$, the negative of ij .
  Thus quaternions have the structure of cross products $ij = k$ for $\hat{x} \times \hat{y} = \hat{z}$ and of (negative) dot products $i^2 = j^2 = k^2 = -1$. A rotation about $\hat{x}$ in the (y, z)-plane:

$$e^{i\phi} j = (1 \cos\phi + i \sin\phi) j = j \cos\phi + k \sin\phi$$

  Other tricks were needed to keep i invariant by the rotation. But this math was too much for the 19th century, so Willard Gibbs and Oliver Heaviside, independently distilled the structure of quaternions into the simpler dot and cross products of vector calculus.
  The use of (i, j, k) as unit vectors is a carry-over from quaternions. Similar contemporary developments were Hermann Grassmann's generalization of cross products to n-dimensions, and William Kingdom Clifford's combination of dot and cross products into a unified "geometric" product, with the same structure as Pauli or Dirac matrices.

ROTATIONS

- Rotations are linear operators (square matrices) that preserve the 'shape' of a set of vectors. They preserve lengths and angles between vectors, thus depends on the metric and are thus called 'orthogonal' transformations. Rotation matrices are also called orthogonal.
  HW#2 explores the similar structure of the imaginary i and the cross product $\times$, and this similar structure is use to 'generate' rotations.
- There are two ways to represent rotations:
  a) Active rotations: physically rotate the vector in a fixed coordinate system for example, to describe a rotating object
  b) Passive rotations: the physical vectors remains fixed, but the reference frame and basis vectors are rotated, giving the vector different components.

  Both types of rotations look the same when transforming components

$$\mathbb{V}' = \mathbb{R}\mathbb{V}$$

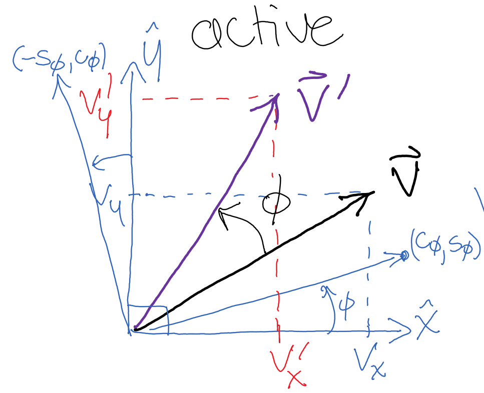
OR

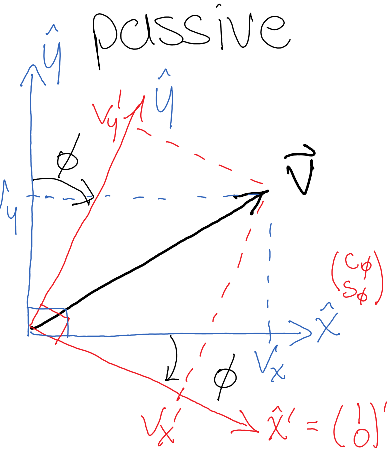
$$V^{i'} = R^{i'}{}_j V^j$$

OR

$$\begin{pmatrix} V'_x \\ V'_y \\ V'_z \end{pmatrix} = \begin{pmatrix} C_\phi & -S_\phi & \\ S_\phi & C_\phi & \\ & & 1 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$

$\mathbb{R}\hat{x}$ $\mathbb{R}\hat{y}$





You can see that the same transformation of components involves rotations in different directions of the vector vs. the axes!

To specify this relation, we need to include basis vectors in the passive formulation:

$$\vec{V} = \underbrace{(\hat{x}\ \hat{y}\ \hat{z})}_{\hat{\mathbb{e}}} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \hat{\mathbb{e}}\,\mathbb{V} = \hat{e}_i V^i = (\hat{x}'\ \hat{y}'\ \hat{z}') \begin{pmatrix} V'_x \\ V'_y \\ V'_z \end{pmatrix} = \hat{\mathbb{e}}'\mathbb{V}' = \hat{e}_{i'} V^{i'}$$

(original basis) (transformed basis)

$$\text{if } \boxed{\mathbb{V}' = \mathbb{R}\mathbb{V}}, \text{ then } \vec{V} = \hat{\mathbb{e}}'\overbrace{\mathbb{V}'}^{} = \underbrace{\hat{\mathbb{e}}'\,\mathbb{R}}_{}\mathbb{V} = \hat{\mathbb{e}}\,\mathbb{V}, \text{ so } \boxed{\hat{\mathbb{e}} = \hat{\mathbb{e}}'\mathbb{R}}$$

Notice the difference in primed vs unprimed and in left vs right multiplication of R. These differences are why v^i are called the contravariant components of $\vec{v}$.

$$(\hat{x}\,\hat{y}\,\hat{z}) = (\hat{x}'\,\hat{y}'\,\hat{z}) \begin{pmatrix} C_\phi & -S_\phi & \\ S_\phi & C_\phi & \\ & & 1 \end{pmatrix}$$

$$\hat{x} = \hat{x}' c_\phi + \hat{y}' s_\phi$$

We can remember this by writing $R^{i'}{}_j$ and $\hat{e}_i$, V^j so there is only one way to write the transformations while still conserving "index height":

$$\hat{e}_{i'} R^{i'}{}_j = \hat{e}_j \qquad V^{i'} = R^{i'}{}_j V^j$$

Let's examine each of these in terms of components in the first column of R:

1) $\hat{\mathbb{e}} = \hat{\mathbb{e}}'\mathbb{R}$ is the linear combination $\hat{x} = \hat{x}' C_\phi + \hat{y}' S_\phi$

2) Alternatively actively transform the components of the vector $\hat{x}' = (\hat{x}'\,\hat{y}'\,\hat{z}') \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$\hat{x} = \mathbb{R}\hat{x}' = \begin{pmatrix} C_\phi & -S_\phi & \\ S_\phi & C_\phi & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} C_\phi \\ S_\phi \\ 0 \end{pmatrix}$$

in the primed system, which is the same as above.

- *Orthogonal transformations are ones that 'preserve the metric' meaning that you get the same value of the dot product of two vectors whether or not you transform them. If this is true for the basis vectors, it will be true for the entire linear space.*

$$g \equiv \hat{e}^T \cdot \hat{e} = (\hat{e}'\mathbb{R})^T(\hat{e}'\mathbb{R}) = \mathbb{R}^T \underbrace{\hat{e}'^T \cdot \hat{e}'}_{g'} \mathbb{R} = \mathbb{R}^T g' \mathbb{R} \overset{\text{orthogonal}}{\equiv} g'$$

If the original basis is orthonormal, ie. $g = \mathbb{I}$ *then the transformed basis is also* $g' = \mathbb{I}$

Thus $\quad g = \mathbb{R}^T g\, \mathbb{R} \quad$ *so* $\quad \mathbb{R}^T\mathbb{R} \,{}^{\times \mathbb{R}^{-1}} = \mathbb{I} \,{}^{\times \mathbb{R}^{-1}} \quad$ *or* $\quad \mathbb{R}^T = \mathbb{R}^{-1}$

Note the role of the transpose is to flip components to form dot products of individual columns.

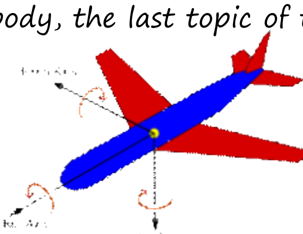
$$\mathbb{R}^T\mathbb{R} = \begin{pmatrix} \boxed{C_\phi \;\; S_\phi} & \\ -S_\phi \;\; C_\phi & \\ & 1 \end{pmatrix}\begin{pmatrix} \boxed{\begin{matrix} C_\phi \\ S_\phi \end{matrix}} & \begin{matrix} -S_\phi \\ C_\phi \end{matrix} & \\ & & 1 \end{pmatrix} = \begin{pmatrix} \boxed{1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\hat{x}\cdot\hat{x} = C_\phi^2 + S_\phi^2 = 1$$
$$\hat{x}\cdot\hat{y} = C_\phi(-S_\phi) + S_\phi C_\phi = 0$$
etc.

- *For orthogonal coordinate systems, the components can be found by taking dot products (the direction cosines)* $\quad R^{i'}{}_j = \hat{e}_{i'} \cdot \hat{e}_j = \cos\theta_{i'j}$

$$R^j{}_{i'} R^{i'}{}_k = \delta^j{}_k.$$

- *As an example, here is a derivation of the transformation matrix for the* [Euler angles]() *of an arbitrary rotation in 3-d. You need 3 parameters: 2 (θ,ϕ) to describe the direction of the new z-axis, and 1 (ψ) to describe the position of the (x,y)-axes rotated about z. These are used in robotics, and also to describe the rotational motion of a rigid body, the last topic of this course. An example is the roll (ψ), pitch (θ), and yaw (ϕ) of an airplaine in a slightly different convention. We obtain the full transformation matrix by composition three elementary rotations of (ϕ,θ,ψ) .*


(wikipedia)

In the standard Euler z-y'-z'' convention, we rotate the coordinate system (ϕ) about the z-axis, then (θ) about the new y-axis, and then (ψ) about the new z-axis. Care is needed to multiply the rotations in the correct order:

a) *Passive:* $\quad 1)\ \hat{e}' = \hat{e}\,\mathbb{R}_{\hat{z}\phi} \quad 2)\ \hat{e}'' = \hat{e}'\,\mathbb{R}_{\hat{y}\theta} \quad 3)\ \hat{e}''' = \hat{e}''\,\mathbb{R}_{\hat{z}\psi}$

$$\left(\left((\hat{e})\mathbb{R}_{\hat{z}\phi} = \hat{e}'\right)\mathbb{R}_{\hat{y}\theta} = \hat{e}''\right)\mathbb{R}_{z\psi} = \hat{e}''' \quad \text{ie} \quad \hat{e}''' = \hat{e}\,\underbrace{\mathbb{R}_{\hat{z}\phi}\mathbb{R}_{\hat{y}\theta}\mathbb{R}_{\hat{z}\psi}}_{\mathbb{R}}$$

a) *Active: this time we must rotate (ψ) first about the z-axis before the z-axis has changed directions, since all rotations are done in the fixed lab frame. Likewise, we must rotate (θ) second while the y-axis is still in its original position, and finally the (ϕ) rotation about z.*

$$\mathbb{R}\mathbb{V} = \mathbb{R}_{\hat{z}\phi}\left(\mathbb{R}_{\hat{y}\theta}\left(\mathbb{R}_{\hat{z}\psi}\mathbb{V}\right)\right) = \left(\underbrace{\mathbb{R}_{\hat{z}\phi}\mathbb{R}_{\hat{y}\theta}\mathbb{R}_{\hat{z}\psi}}_{\mathbb{R}}\right)\mathbb{V}$$

Either way, the full rotation matrix is the same:

$$\mathbb{R} = \begin{pmatrix} C_\phi & -S_\phi & \\ S_\phi & C_\phi & \\ & & 1 \end{pmatrix} \begin{pmatrix} C_\theta & & S_\theta \\ & 1 & \\ -S_\theta & & C_\theta \end{pmatrix} \begin{pmatrix} C_\psi & -S_\psi & \\ S_\psi & C_\psi & \\ & & 1 \end{pmatrix} = \begin{pmatrix} C_\psi C_\theta S_\phi - S_\psi S_\phi & -S_\psi C_\theta C_\phi - C_\psi S_\phi & S_\theta C_\phi \\ C_\psi C_\theta S_\phi - S_\psi C_\phi & -S_\psi C_\theta S_\phi + C_\psi C_\phi & S_\theta S_\phi \\ -C_\psi S_\theta & S_\psi S_\theta & C_\theta \end{pmatrix}$$

$-S_\theta$ C_θ

S_θ

$$\begin{pmatrix} C_\theta C_\phi & -S_\phi & S_\theta C_\phi \\ C_\theta S_\phi & C_\phi & S_\theta S_\phi \\ -S_\theta & 0 & C_\theta \end{pmatrix}$$

wikipedia:

$$\underset{\phi\quad\theta\quad\psi}{Z_1 Y_2 Z_3} = \begin{array}{c} \hat{x} \\ \hat{y} \\ \hat{z} \end{array} \begin{bmatrix} c_1 c_2 c_3 - s_1 s_3 & -c_3 s_1 - c_1 c_2 s_3 & c_1 s_2 \\ c_1 s_3 + c_2 c_3 s_1 & c_1 c_3 - c_2 s_1 s_3 & s_1 s_2 \\ \underbrace{-c_3 s_2}_{\hat{e}_1'''} & \underbrace{s_2 s_3}_{\hat{e}_2'''} & \underbrace{c_2}_{\hat{e}_3'''} \end{bmatrix}$$

active $Z_\phi Y'_\theta Z''_\psi$

passive $Z_\psi - Y_\theta - Z_\phi$