L08 Stretches: Eigensystems

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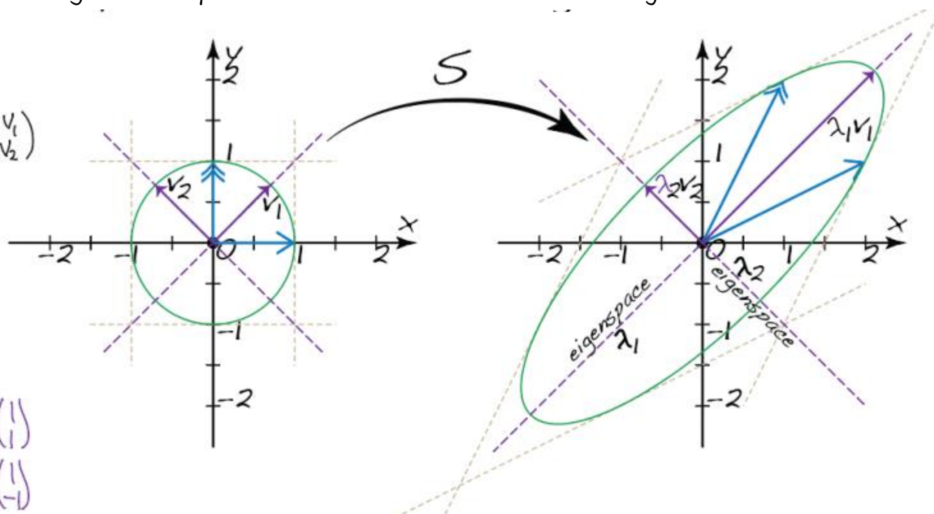
Last lecture, we saw that rotations were special linear transformations where $R^T = R^{-1}$
Now we explore 'opposite' transformations" stretches, where $S^T = S$
The polar decomposition theorem states that any square matrix can be decomposed
into the product of a stretch and a rotation: $M = RS$ where $R^T = R^{-1}$ $S^T = S$.
or in the opposite order: $M = S'R$ where $S' = R^T S R$
in analogy with complex numbers: $z = \rho e^{i\phi} = e^{\alpha + i\phi} = e^{w}$ $(\rho \equiv e^{\alpha})$
Just as complex numbers $z = x + iy$ separate
into real $\alpha = \frac{1}{2}(w + w^*) = \alpha^*$ and imaginary $i\phi = \frac{1}{2}(w - w^*) = (-i\phi)^*$ parts,
Matrices $G = T + A$ separate into symmetric $T = \frac{1}{2}(G + G^T) = T^T$
and antisymmetric $A = \frac{1}{2}(G - G^T) = -A^T$ parts. This comparison between matrices and complex numbers runs deep in the normal matrix analogy .
However, the polar decomposition theorem cannot be proved using the exponential because matrix multiplication doesn't commute (Baker Campbell Hausdorff formula).

The geometry of the eigenvalue equation $M\vec{v} = \vec{v}\lambda$ is that M stretches $\vec{v}$ by the factor λ .

$$S v = \lambda v$$
$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$



Algebraically,

$$S\vec{v} = \lambda I \vec{v} \qquad \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (\lambda - 2)^2 - (1)^2 = 0 \qquad \lambda - 2 = \pm 1, \quad \lambda = 3, 1$$

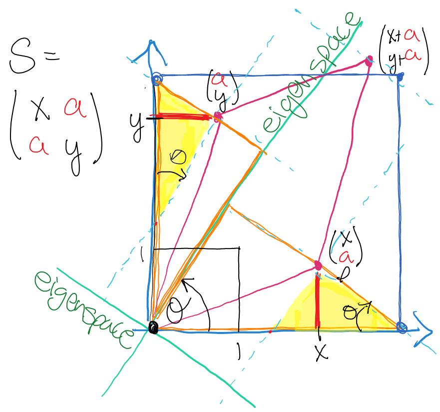
$$(S - \lambda I)\vec{v} = 0 \qquad \lambda_1 = 3: \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$|S - \lambda I| = 0$$

(characteristic eq'n)

$$\lambda_2 = 1: \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Geometric explanation why a symmetric 2-d matrix has no rotation, just pure stretch.



1. A symmetric matrix transforms $\hat{x}, \hat{y}$ to the red dots, the same distance a (thick red line) from both axes. A rotation would have made one distance larger and the other smaller.
2. Form two triangles (yellow) from the red vertices to the x, y-axes at the same angle θ, such that the rectangle at the outer vertices (blue) is a square.
3. The orange triangles are the same size and shape as (congruent) to the yellow triangles, so shrinking the blue square along the green line transforms it into the trapezoid image of S (blue → red dots).

Algebraic explanation why any symmetric matrix has orthogonal eigenvectors--two principles:

1) S "has eyes in the back of its head"! Example symmetry: "A man, a plan, a canal: Panama"
(matrices multiply in both directions)

$$\begin{pmatrix} x & x \end{pmatrix} = \begin{pmatrix} x & x \end{pmatrix}\begin{pmatrix} x & x \\ x & x \end{pmatrix} \qquad \begin{pmatrix} x & x \\ x & x \end{pmatrix}\begin{pmatrix} x \\ x \end{pmatrix} = \begin{pmatrix} x \\ x \end{pmatrix} \qquad S\vec{v} = \vec{w}$$

$$\left(\vec{y}^T = \vec{u}^T S\right)^T \Rightarrow S^T\vec{u} = \vec{y}$$

2) A "matrix sandwich" does both at the same time in either order (associative), unlike English, for example "Bald Eagle Scout"

$$(\vec{u}^T S)\cdot\vec{v} = \vec{u}^T S \vec{v} = \vec{u}^T\cdot(S\vec{v})$$

Apply these principles to eigenvectors of a symmetric matrix:

$$\lambda_i^*(\vec{v}_i\cdot\vec{v}_j) = \vec{v}_i^\dagger S \vec{v}_j = (\vec{v}_i\cdot\vec{v}_j)\lambda_j \Rightarrow \vec{v}_i\cdot\vec{v}_j(\lambda_i^* - \lambda_j) = 0$$

1) if $i = j$ then $\|v_i\|^2 \neq 0 \Rightarrow \lambda_i^* = \lambda_i \in \mathbb{R}$ (eigenvalues are real)

2) if $\lambda_i \neq \lambda_j$ then $\vec{v}_i\cdot\vec{v}_j = 0$ (orthogonal eigenvectors)

3) if $\lambda_i = \lambda_j = \ldots \equiv \lambda$ let $\vec{v} = \alpha\vec{v}_i + \beta\vec{v}_j + \ldots \;\forall\; \alpha, \beta, \ldots \in \mathbb{R}$

then $S\vec{v} = \alpha S\vec{v}_1 + \beta S v_2 + \ldots = \alpha\vec{v}_1\lambda + \beta\vec{v}_2\lambda + \ldots = \vec{v}\lambda$

($\vec{v}_1, \vec{v}_2 \ldots$ span a whole λ-eigenspace! — choose an orthonormal basis)

Transformation matrix V of eigenvectors and different interpretations of eigenvectors:

0) The eigenvalue equation can be augmented with all eigenvectors

$$S\vec v_1 = \vec v_1 \lambda_1, \quad S\vec v_2 = \vec v_2 \lambda_2 \;\Rightarrow\; S(\vec v_1 \mid \vec v_2 \mid \cdots) = (\vec v_1\lambda_1 \mid \vec v_2 \lambda_2 \mid \cdots) = (\vec v_1 \mid \vec v_2 \mid \cdots)\, D$$

$$\begin{pmatrix} x & x \\ x & x \end{pmatrix}\begin{pmatrix} v_1^x & v_2^x & \cdots \\ v_1^y & v_2^y & \end{pmatrix} = \begin{pmatrix} v_1^x & v_2^x & \cdots \\ v_1^y & v_2^y & \end{pmatrix}\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$$

$$\boxed{SV = VD}$$ (diagonal D must multiply from the right)

1) V is a transformation matrix into the eigenbasis, the frame where S' is a diagonal matrix

$$V^T V = \begin{pmatrix} \vec v_1 \\ \vec v_2 \\ \vdots \end{pmatrix} \cdot (\vec v_1 \mid \vec v_2 \mid \cdots) = \begin{pmatrix} v_1^x & v_1^y & \cdots \\ v_2^x & v_2^y & \cdots \\ \vdots & \vdots & \end{pmatrix}\begin{pmatrix} v_1^x & v_2^x & \cdots \\ v_1^y & v_2^y & \cdots \\ \vdots & \vdots & \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

"orthogonality"

$$\vec v_i \cdot \vec v_j = \delta_{ij}$$ (V is an orthogonal matrix – rotation)

$$V^T(SV = VD) \qquad \begin{pmatrix} \vec v_1 \\ \vec v_2 \\ \vdots \end{pmatrix} \cdot S(\vec v_1 \mid \vec v_2 \mid \cdots) = \begin{pmatrix} \vec v_1 \\ \vec v_2 \\ \vdots \end{pmatrix} \cdot (\vec v_1 \lambda_1 \mid \vec v_2 \lambda_2 \mid \cdots) = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$$

$$V^T S V = D$$

$$V\vec x' = \underbrace{(\vec v_1 \mid \vec v_2 \mid \cdots)}_{\text{original components}}\begin{pmatrix} x_1' \\ x_2' \\ \vdots \end{pmatrix} = \vec x \qquad \vec y = S\vec x$$

$$V\vec y' = \vec y \qquad V\vec y' = S V \vec x'$$

$$\boxed{V^T V = I} \qquad \boxed{S' = V^T S V = D} \qquad \vec y' = \underbrace{V^T S V}_{S'} \vec x' = D\vec x'$$

Example: equation of an ellipse

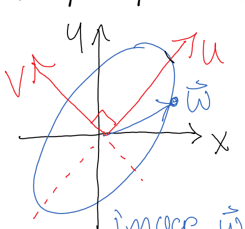


image $\vec w = S\vec z$ of the circle $\vec z \cdot \vec z = 1$

$$1 = \left(\frac{u}{\lambda_1}\right)^2 + \left(\frac{v}{\lambda_2}\right)^2 = (u\;v)\begin{pmatrix} 1/\lambda_1^2 & \\ & 1/\lambda_2^2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \vec w'^T D^{-2} \vec w'$$

$$= \vec w'^T (V^T S^{-1} \cancel{V})(\cancel{V^T} S^{-1} V)\vec w'$$

$$= (V\vec w')^T S^{-2} (V\vec w')$$

$$= \vec w^T S^{-2} \vec w = \vec z \cdot \vec z$$

$$= a x^2 + 2bxy + c y^2$$

2) *The spectral decomposition or eigen-decomposition of the matrix operator S*

$$VV^T = (\vec{v}_1 | \vec{v}_2 | \cdots) \begin{pmatrix} \vec{v}_1 \cdot \\ \vec{v}_2 \cdot \\ \vdots \end{pmatrix} = \vec{v}_1\vec{v}_1\cdot + \vec{v}_2\vec{v}_2\cdot + \ldots \equiv P_1 + P_2 + \ldots = I$$

"closure" - completeness
$$\begin{pmatrix}1&&\\&0&\\&&0\end{pmatrix} + \begin{pmatrix}0&&\\&1&\\&&0\end{pmatrix} + \ldots = \begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix}$$

$$(SV = VD)V^T \qquad (\vec{v}_1 | \vec{v}_2 | \cdots) \begin{pmatrix}\lambda_1&&\\&\lambda_2&\\&&\lambda_3\end{pmatrix} \begin{pmatrix} \vec{v}_1 \cdot \\ \vec{v}_2 \cdot \\ \vdots \end{pmatrix} = \vec{v}_1\lambda_1\vec{v}_1\cdot + \vec{v}_2\lambda_2\vec{v}_2\cdot + \ldots$$

$$S = VDV^T \qquad = \lambda_1 P_1 + \lambda_2 P_2 + \ldots = S$$

$$\boxed{I = VV^T = \sum_i P_i} \qquad \boxed{M = VDV^T = \sum_i \lambda_i P_i}$$

The Singular Value Decomposition is Polar Decomposition followed by Eigen-Decomposition
It is defined for any matrix, not just a square matrix, and is useful in cases for linear functions transforming one vector space into a different one, (as opposed to operators on the same space).

$$M = RS = R(VWV^T) = (RV)WV^T = UWV^T \qquad W = \begin{pmatrix}\lambda_1&&\\&\lambda_2&\\&&\ddots\end{pmatrix}$$

$$\text{where } \boxed{U^TU} = (RV)^T(RV) = V^T\overbrace{R^TR}^{I}V = \boxed{V^TV = I}$$

It is a sum of outer products with stretch factors $\boxed{M = UWV^T = \sum_i \vec{u}_i \lambda_i \vec{v}_i \cdot}$

```
octave> A=[2 1;1 2];
octave> [V,D]=eig(A)
V = -0.70711   0.70711    D = 1   0
     0.70711   0.70711        0   3

octave> V'*V
   1.0000e+00   2.2371e-17
   2.2371e-17   1.0000e+00
```

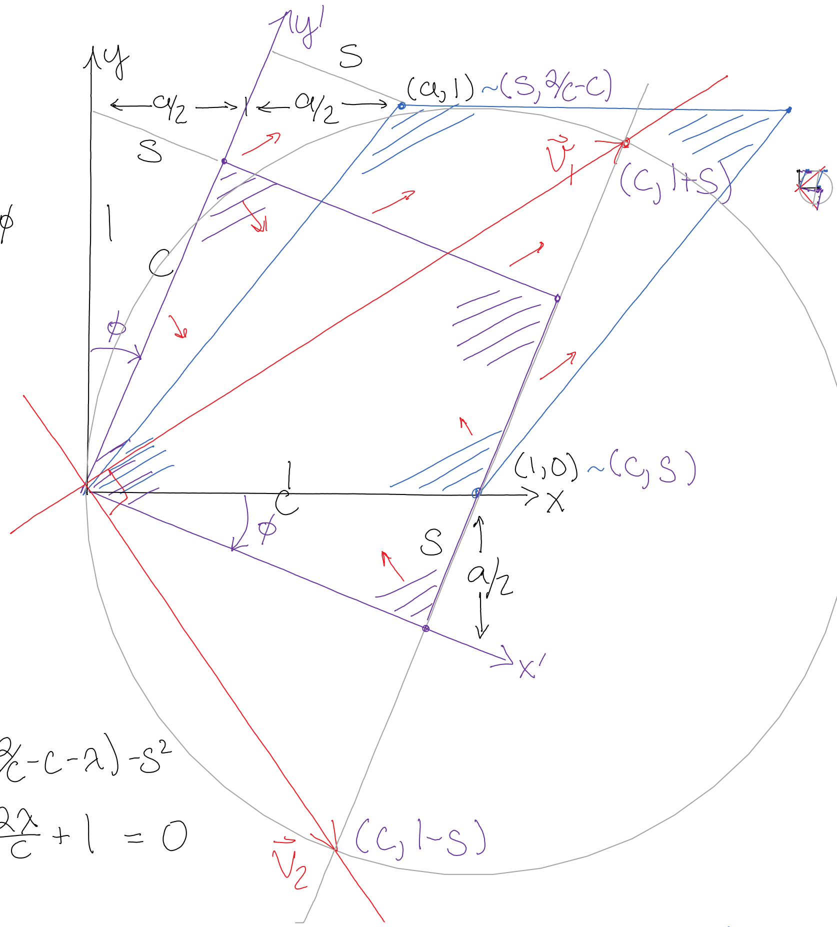
```
octave> V'*A*V
   1.0000e+00   2.2371e-17
  -4.3909e-17   3.0000e+00

octave> V*D*V'
   2.00000   1.00000
   1.00000   2.00000
```

```
octave> M=[1 2;0 1];

octave> [U,W,V]=svd(M)

U = 0.92388  -0.38268
    0.38268   0.92388

W = 2.41421         0
          0   0.41421

V = 0.38268  -0.92388
    0.92388   0.38268
```

```
octave> U'*U
   1.0000e+00  -1.7409e-17
  -1.7409e-17   1.0000e+00

octave> V'*V
   1.0000e+00  -8.6102e-18
  -8.6102e-18   1.0000e+00

octave> U*W*V'
   1.0000e+00   2.0000e+00
   8.6102e-18   1.0000e+00

octave> U'*M*V
   2.4142e+00   9.1395e-17
   7.1471e-17   4.1421e-01
```

Example: decomposition of a shear matrix

$$A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = RS$$



$$R = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \qquad R^T R = I \qquad c = \cos\phi \quad s = \sin\phi$$

$$S = R^T A = \begin{pmatrix} c & ca - s \\ s & sa + c \end{pmatrix}$$

$$S^T = S \qquad ca = 2s$$

$$\sqrt{a^2+4} \quad \phi \quad a \quad 2 \qquad a/2 = \tan\phi$$

$$sa + c = \frac{2s^2}{c} + c = \frac{2}{c} - c$$

$$S = \begin{pmatrix} c & s \\ s & 2/c - c \end{pmatrix}$$

$$\begin{vmatrix} c - \lambda & s \\ s & 2/c - c - \lambda \end{vmatrix} = (c-\lambda)(2/c - c - \lambda) - s^2$$

$$= \lambda^2 - \frac{2\lambda}{c} + 1 = 0$$

$$\lambda_\pm = \frac{1}{c} \pm \sqrt{\frac{1}{c^2} - 1} = \frac{1 \pm s}{c}$$

$$\left[\begin{pmatrix} c - \left(\frac{1\pm s}{c}\right) & s \\ s & \frac{2}{c} - c - \left(\frac{1 \pm s}{c}\right) \end{pmatrix} = \begin{pmatrix} -\frac{s(s\pm 1)}{c} & s \\ s & \frac{s(s \mp 1)}{c} \end{pmatrix}\right] \begin{pmatrix} c \\ s \pm 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad V = \begin{pmatrix} c & c \\ s+1 & s-1 \end{pmatrix}$$

(unnormalized)

thus

$$A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = RS = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}\begin{pmatrix} c & s \\ s & 2/c - c \end{pmatrix} = \frac{1}{\sqrt{a^2+4}}\begin{pmatrix} 2 & a \\ -a & 2 \end{pmatrix} \cdot \frac{1}{\sqrt{a^2+4}}\begin{pmatrix} 2 & a \\ a & a^2+2 \end{pmatrix}$$

$$\text{where } S = VWV^T = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{c}{\sqrt{1+s}} & \frac{c}{\sqrt{1-s}} \\ \sqrt{1+s} & -\sqrt{1-s} \end{pmatrix} \cdot \begin{pmatrix} \frac{1+s}{c} & \\ & \frac{1-s}{c} \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{c}{\sqrt{1+s}} & \sqrt{1+s} \\ \frac{c}{\sqrt{1-s}} & -\sqrt{1-s} \end{pmatrix} = \begin{pmatrix} c & s \\ s & 2/c - c \end{pmatrix}$$

$$SV = \begin{pmatrix} c & s \\ s & 2/c - c \end{pmatrix}\begin{pmatrix} c & c \\ s+1 & s-1 \end{pmatrix} = \begin{pmatrix} 1+s & 1-s \\ (1+s)^2/c & -(1-s)^2/c \end{pmatrix} = \begin{pmatrix} c & c \\ s+1 & s-1 \end{pmatrix}\begin{pmatrix} \frac{1+s}{c} & \\ & \frac{1-s}{c} \end{pmatrix} = VW$$

$$A = \underbrace{U}_{RV} W V^T = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{1+s} & \sqrt{1-s} \\ \frac{c}{\sqrt{1+s}} & -\frac{c}{\sqrt{1-s}} \end{pmatrix} \cdot \begin{pmatrix} \frac{1+s}{c} & \\ & \frac{1-s}{c} \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{c}{\sqrt{1+s}} & \sqrt{1+s} \\ \frac{c}{\sqrt{1-s}} & -\sqrt{1-s} \end{pmatrix} = \begin{pmatrix} 1 & \frac{2s}{c} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$