L14 Hamiltonian

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* Lagrange equations minimize the "action" $S \equiv \int_a^b \mathcal{L}\,dt$

$$\delta S = \int_a^b dt\, \delta\mathcal{L}(\dot q, q, t) = \int_a^b dt \left[\frac{\partial \mathcal{L}}{\partial \dot q}\delta \dot q + \frac{\partial \mathcal{L}}{\partial q}\delta q + \cancel{\frac{\partial \mathcal{L}}{\partial t}\delta t}\right]$$

$$= \cancel{\frac{\partial \mathcal{L}}{\partial \dot q}\delta q\Big|_a^b} + \int_a^b dt \underbrace{\left[-\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot q} + \frac{\partial \mathcal{L}}{\partial q}\right]}_{0}\delta q(t) \qquad \text{[Hamilton's principle]}$$

- if $\frac{\partial \mathcal{L}}{\partial q} = 0$, then $\frac{\partial \mathcal{L}}{\partial \dot q}$ is constant ("first integral of motion")
- define $\boxed{p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot q^i}}$ as the "generalized momentum"
- then $\boxed{\dot p_i = \frac{\partial \mathcal{L}}{\partial q^i}}$ is Lagrange's equation [note the symmetry!]
- thus $d\mathcal{L}(\dot q, q, t) = p\,d\dot q + \dot p\,dq + \frac{\partial \mathcal{L}}{\partial t}dt = \frac{d}{dt}\underbrace{p\,dq}_{dS_0} + \frac{\partial \mathcal{L}}{\partial t}dt$

* Hamilton equations: do a change of variables ("Legendre transform")

to make the equations symmetric w/r $q \leftrightarrow p$, not $q \leftrightarrow \dot q$

- use $d(p\dot q) = dp\cdot \dot q + p\,d\dot q$ to switch $d\dot q$ to dp

$$d(\underbrace{p\dot q - \mathcal{L}}_{\mathcal{H}}) = \boxed{\dot q}\,dp \quad \boxed{-\dot p}\,dq \quad \boxed{-\frac{\partial \mathcal{L}}{\partial t}}\,dt$$

$$d\mathcal{H}(p,q,t) = \boxed{\frac{\partial \mathcal{H}}{\partial p}}dp + \boxed{\frac{\partial \mathcal{H}}{\partial q}}dq + \boxed{\frac{\partial \mathcal{H}}{\partial t}}dt$$

convert $\dot q \to p$!
$\frac{1}{2}mv^2 \to \frac{p^2}{2m}$

where the "Hamiltonian" is defined $\mathcal{H}(p,q,t) \equiv p\dot q - \mathcal{L}(\dot q, q, t)$

- this results in the following "Hamilton equations" of motion

$$\boxed{\dot q = \frac{\partial \mathcal{H}}{\partial p} \quad \dot p = -\frac{\partial \mathcal{H}}{\partial q}} \quad \text{OR} \quad \frac{d}{dt}\begin{pmatrix}p\\q\end{pmatrix} = \underbrace{\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}}_{M}\begin{pmatrix}\partial_p\\ \partial_q\end{pmatrix}\mathcal{H} \quad \text{OR} \quad \boxed{\dot a = M\nabla_a \mathcal{H}}$$

HO2 #2

"symplectic form" $M = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$ vs. "symmetric metric" $g = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$

* Examples: 1) if $\mathcal{L} = T - V = \frac{1}{2}mv^2 + V$ then $p = \frac{\partial T}{\partial v} = mv$

$$\mathcal{H} = pv - \mathcal{L} = mv^2 - \tfrac{1}{2}mv^2 + V = \frac{p^2}{2m} + V$$

- note we had to substitute $v = p/m$ to get $\mathcal{H}(p,x,t)$
- in quantum mechanics, $p \to -i\hbar\frac{\partial}{\partial x}$ so $\mathcal{H}\psi = \left[\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\psi(x)$, part of the "Schrödinger equation" $\mathcal{H}\psi = E\psi$ where $E \to i\hbar\frac{\partial}{\partial t}$

2) if $T = \frac{1}{2}m(\rho\dot{\phi})^2$ then $p_\phi = \frac{\partial}{\partial\dot{\phi}}T = \underbrace{m\rho^2}_{I}\dot{\phi} = mvr$ "angular momentum"

then $\mathcal{H} = \frac{p_\phi^2}{2\underbrace{m\rho^2}_{I}} + V(\phi)$, which is rotational energy. $I \equiv m\rho^2$

* Action-Angle: For a "harmonic oscillator"

$\vec{F} = -k\vec{x}$ $V = \frac{1}{2}kx^2$, $a \equiv \binom{p}{q}$ traces out

a circle in "phase space" $\{a = (p,x)\}$

See [H04#2]