L15 Conservation

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https://www.sciencenews.org/article/physicists-have-narrowed-the-mass-range-for-hypothetical-dark-matter-axions

* Summary of Lagrangian, Action, Hamiltonian:

$$\delta\Big[\underbrace{S\equiv\int_a^b \mathcal{L}\,dt}_{\text{action}}\Big] = \int_a^b dt\left[\frac{\partial\mathcal L}{\partial\dot q}\delta\dot q + \frac{\partial \mathcal L}{\partial q}\delta q + \frac{\partial\mathcal L}{\partial t}\delta t\right] = \frac{\partial\mathcal L}{\partial\dot q}\delta q\Big|_a^b + \int_a^b dt\Big[\underbrace{-\frac{d}{dt}\frac{\partial\mathcal L}{\partial \dot q}}_{\dot p} - \frac{\partial\mathcal L}{\partial q}\Big]\delta q \qquad \delta S/\delta q(t) = 0$$

$$\frac{\partial\mathcal L}{\partial\dot q} = p, \qquad \frac{\partial\mathcal L}{\partial q} = \dot p, \qquad \frac{\partial\mathcal L}{\partial t} = \frac{\partial\mathcal L}{\partial t}$$

$$d\underbrace{\mathcal L(\dot q,q,t)}_{\text{Lagrangian}} = p\,d\dot q + \dot p\,dq + \frac{\partial\mathcal L}{\partial t}dt \qquad \text{Lagrange eq.}$$

$$d(\underbrace{\mathcal H = p\dot q - \mathcal L}_{\text{Hamiltonian}}) = \dot q\,dp + -\dot p\,dq + -\frac{\partial\mathcal L}{\partial t}dt \qquad \underbrace{d(p\dot q) = dp\,\dot q + p\,d\dot q}_{\text{(Legendre)}}$$

$$\underbrace{d\mathcal H(p,q,t)}_{\text{Convert } \dot q \to p.} = \frac{\partial\mathcal H}{\partial p}dp + \frac{\partial \mathcal H}{\partial q}dq + \frac{\partial\mathcal H}{\partial t}dt \qquad \text{Hamilton eqs.}$$

$$\dot q = \frac{\partial\mathcal H}{\partial p}, \qquad -\dot p = \frac{\partial\mathcal H}{\partial q}, \qquad -\frac{\partial\mathcal L}{\partial t} = \frac{\partial\mathcal H}{\partial t}$$

⇒ chain & product rules!

* Examples of conserved quantities:

- we have already seen $\dot p_i = \frac{\partial\mathcal L}{\partial q^i} = -\frac{\partial\mathcal H}{\partial q^i}$ so that if $\mathcal L$ or $\mathcal H$ is symmetric w/r q^i, ie $\frac{\partial\mathcal L}{\partial q^i} = 0$ then p_i is conserved: $\frac{dp_i}{dt} = 0$.
- likewise, $\dot{\mathcal H} = \left(d\mathcal H = \dot p\,dq - \dot q\,dp - \frac{\partial\mathcal L}{\partial t}\right)/dt = \cancel{\dot p\dot q} - \cancel{\dot q\dot p} - \frac{\partial\mathcal L}{\partial t} = \frac{\partial\mathcal H}{\partial t}$, so if $\mathcal L$ (or $\mathcal H$) is symmetric w/r time, then $\mathcal H = T+V = E$ (but not $\mathcal L$) is conserved.

* Noether's first theorem: https://arxiv.org/pdf/physics/9807044.pdf

Given a symmetry $\mathcal L(s) = \mathcal L(\dot q(s), q(s), t)$ with $\mathcal L' = \dot F$, where $' \equiv \frac{\partial}{\partial s}$

Let $I \equiv p_i q'^i - F$ (the Noether conserved current).

Then $\dot I = \left(\frac{d}{dt}(p_i q'^i) = p_i \dot q'^i + \dot p_i q'^i = \frac{\partial\mathcal L}{\partial\dot q^i}\frac{d\dot q^i}{ds} + \frac{\partial\mathcal L}{\partial q^i}\frac{dq^i}{ds} = \mathcal L'\right) - \dot F = 0.$

- Examples of Noether's theorem:

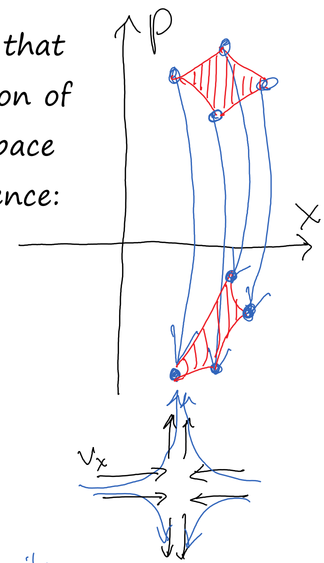
Conservation of:

a) if $\frac{\partial \mathcal{L}}{\partial q^i} = 0$ then let $q^i(s) = q^i_0 + s$ $\quad \dot{q}(s) = \dot{q}_0 \quad I = p_i$ "momentum"

b) if $\frac{\partial \mathcal{L}}{\partial t} = 0$ then let $s = t \quad F = \mathcal{L} \quad I = p_i \dot{q}^i - \mathcal{L} = \mathcal{H}$ "energy"

c) if $\frac{\partial \mathcal{L}}{\partial x} = -Q$ then let $s = x \quad F = -Q \quad I = p_i q'^i - F = Q$ "charge"

* Liouville's theorem: https://en.wikipedia.org/wiki/Liouville%27s_theorem_(Hamiltonian)



Hamilton's eq's describe a flow in phase space. Liouville's theorem asserts that it is incompressible. This is another conservation principle, the conservation of phase space volume, and the particles inside the volume evolve in phase space according to Hamilton's equations. An incompressible fluid has no divergence:

$$\nabla \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \qquad \text{"net outward velocity"}$$

The actual outward flow of particles is called the flux density $\vec{J} = \rho \vec{v}$

$$\text{where} \quad \underbrace{J_x}_{\text{flux density}} \equiv \underbrace{\frac{dI}{da_x}}_{\text{current / area}} = \underbrace{\frac{dQ}{dy\,dz\,dt}}_{\text{"density in spacetime"}} = \underbrace{\frac{dQ}{dx\,dy\,dz}}_{\text{charge / volume}} \cdot \underbrace{\frac{dx}{dt}}_{\text{dist / time}} = \underbrace{\rho}_{\text{charge density}} \cdot \underbrace{v_x}_{\text{velocity}}$$

$$\text{so} \quad I = \oint \vec{J} \cdot d\vec{a} = \int \nabla \cdot \vec{J}\, d\tau \quad [\text{flux}] = -\frac{dQ}{dt} = \int -\frac{\partial \rho}{\partial t}\, d\tau$$

$$\text{Thus,} \quad \underbrace{-\frac{\partial \rho}{\partial t}}_{\text{local losses}} = \underbrace{\nabla \cdot \vec{J}}_{\text{outward flux}} = \nabla \cdot (\rho \vec{v}) = \underbrace{(\vec{v} \cdot \nabla)\rho}_{\text{directional derivative}} + \rho \underbrace{(\nabla \cdot \vec{v})}_{\text{divergence}}$$

We relate this to $\nabla \cdot \vec{v}$ by taking the derivative along the trajectory (convective derivative) by using the chain rule for partial derivatives:

$$\underbrace{\dot{\rho}}_{\text{gain along trajectory}} = \underbrace{\frac{\partial \rho}{\partial t} + \frac{dx}{dt} \cdot \frac{\partial \rho}{\partial x} + \frac{dy}{dt} \cdot \frac{\partial \rho}{\partial y} + \dots}_{\text{chain rule for partial derivatives}} = \underbrace{\left[\frac{\partial}{\partial t} + \vec{v} \cdot \nabla\right]\rho}_{\text{convective derivative}} \underset{\text{(from above)}}{=} \underbrace{-\rho\, \nabla \cdot \vec{v}}_{\text{inward flux along trajectory}}$$

Thus the convective derivative of density only depends on the divergence of the velocity. The same applies to the density in phase space (p,x), where the divergence is always zero:

$$\nabla_a \cdot \dot{a} = \nabla_a \cdot \mathbb{M} \nabla_a \mathcal{H} = 0 \qquad a \equiv \begin{pmatrix} p \\ q \end{pmatrix}$$ (symplectic structure of $\mathbb{M}$)

$$= \frac{\partial}{\partial q^i}\dot{q}^i + \frac{\partial}{\partial p_i}\dot{p}_i = \frac{\partial}{\partial q^i}\frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial}{\partial p_i}\frac{\partial \mathcal{H}}{\partial q^i} = 0$$ (equality of mixed partials)

thus, $$\dot{\rho} = \frac{\partial \rho}{\partial t} + \dot{a}\cdot\nabla_a \rho = -\rho\,\nabla_a \cdot \dot{a} = 0$$ [Liouville's theorem]

Similar concepts (chain rule, Hamilton's equations) can be used to calculate the convective derivative on any function $f(p,q,t)$ along the trajectory of a particle:

$$\dot{f} = \frac{\partial f}{\partial t} + \underbrace{\frac{\partial f}{\partial q^i}\frac{dq^i}{dt} + \frac{\partial f}{\partial p_i}\frac{dp_i}{dt}}_{\frac{\partial f}{\partial q^i}\frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial \mathcal{H}}{\partial q^i} \equiv \{f, \mathcal{H}\}} = \frac{\partial f}{\partial t} + \{f, \mathcal{H}\}$$

(Poisson bracket)

This is the classical mechanical analog of the commutator in quantum mechanics. $[\hat{f}, \hat{H}]$

In terms of the Poisson bracket, Liouville's theorem is written $\{\rho, \mathcal{H}\} = -\frac{\partial \rho}{\partial t}$