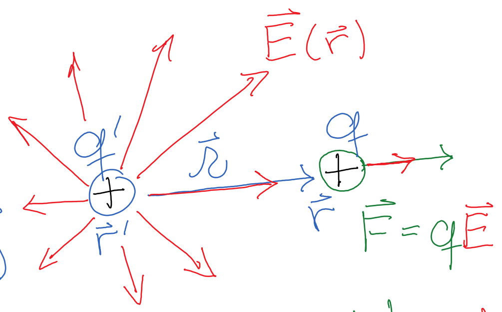
L16 Electromagnetism

Saturday, September 19, 2020 07:34

* Coulomb's law:

$$\vec F = \frac{1}{4\pi\varepsilon}\cdot\frac{qq'}{r^2}\hat r = q\underbrace{\int\frac{\overset{dq'}{\rho' d\tau'}\,\hat r}{4\pi\varepsilon_0 r^2}}_{\vec E(\vec r)}$$

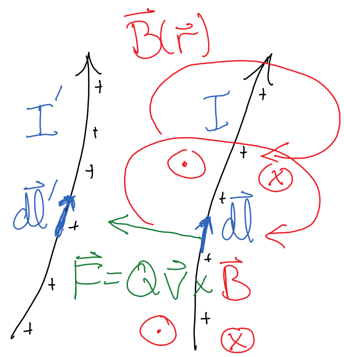


$W = qV$

$$\begin{cases} \nabla\times\vec E = 0 \;\Rightarrow\; \vec E = -\nabla V \\ \qquad\qquad\qquad\quad \Downarrow \\ \nabla\cdot\vec D = \rho \;\Rightarrow\; -\nabla\cdot\varepsilon\nabla V = \rho \end{cases} \qquad \vec D \equiv \varepsilon\vec E$$

* Biot-Savart law:

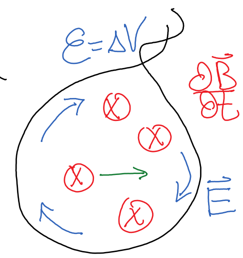
$$\vec F = -\frac{\mu}{4\pi}\iint I d\vec l\cdot I' d\vec l'\,\frac{\hat r}{r^2} = \underbrace{q\vec v}_{\int I d\vec l} \times \underbrace{\int\frac{\overset{I' d\vec l'}{\vec J' d\tau'}\times\hat r}{4\pi\mu\, r^2}}_{\vec B(\vec r)}$$



$$\begin{cases} \nabla\cdot\vec B = 0 \;\Rightarrow\; \vec B = \nabla\times\vec A \\ \qquad\qquad\qquad\quad \Downarrow \\ \nabla\times\vec H = \vec J \;\Rightarrow\; \nabla\times\frac{1}{\mu}\nabla\times\vec A = \vec J \end{cases} \qquad \vec B = \mu\vec H \qquad \nabla\cdot\vec J = 0$$

* Faraday's law:

$$\mathcal{E}_E = -\frac{d}{dt}\Phi_B \qquad \oint_{\partial S}\vec E\cdot d\vec l = \int_S \nabla\times\vec E\cdot d\vec a = \int_S -\frac{\partial\vec B}{\partial t}\cdot da$$



$$\left\{ \nabla\times\vec E + \frac{\partial\vec B}{\partial t} = \nabla\times\left(\vec E + \frac{\partial\vec A}{\partial t}\right) = 0 \;\Rightarrow\; \vec E = -\nabla V - \frac{\partial\vec A}{\partial t} \right.$$

* Maxwell's displacement current:

$$\left\{ \nabla\times\vec H = \vec J + \left(\vec J_D \equiv \frac{\partial\vec D}{\partial t}\right) \right.$$

$$\Rightarrow \nabla\cdot\vec J = \nabla\cdot\left(\cancel{\nabla}\times\vec H - \overset{\nabla\cdot\vec D = \rho}{\frac{\partial\vec D}{\partial t}}\right) = -\frac{\partial\rho}{\partial t} \quad \text{"Continuity Eq'n"}$$

* Gauge invariance:

if $V \to V + \frac{\partial\chi}{\partial t}$, $\vec A \to \vec A - \nabla\chi$ then $(\vec E,\vec B) \to (\vec E,\vec B)$

$\nabla\times\vec A = \vec B$, but $\nabla\cdot A = ?$ (gauge) determines $\vec A$ up to $\nabla^2\chi$

SUMMARY:

$$\underbrace{\Delta \vec V = \frac{\partial \chi}{\partial t} \quad \Delta \vec E = \vec 0 \qquad \Delta \vec A = -\nabla \chi \quad \Delta \vec B = \vec 0}_{\text{gauge transform}}$$

$$\underbrace{\vec E = -\nabla V - \frac{\partial \vec A}{\partial t} \qquad \vec B = \nabla \times \vec A}_{\text{potentials}}$$

$$\nabla \times \vec E + \frac{\partial \vec B}{\partial t} = 0$$

$$\nabla \cdot \vec B = 0 \qquad \vec B = \mu \vec H$$

$$\nabla \cdot \vec D = \rho \qquad \vec D = \varepsilon \vec E$$

$$\underbrace{\vec F = q(\vec E + \vec v \times \vec B)}_{\text{Lorentz force}} \qquad \underbrace{\frac{\partial \rho}{\partial t} + \nabla \cdot \vec J = 0}_{\text{continuity}} \qquad \underbrace{\nabla \times \vec H - \frac{\partial \vec D}{\partial t} = \vec J}_{\text{Maxwell}} \underbrace{= \sigma \vec E}_{\text{constitutive}}$$

MAP:

$$\begin{tikzcd}
\text{mechanics:} & & (W, \vec p) \arrow[r, "d"] & \vec F \arrow[r, "d"] & \vec 0 & \\
\text{Noether} & & & & & \\
\text{force eq's:} & \chi \arrow[r, "d"] & (V, \vec A) \arrow[uu, "q"] \arrow[r, "d"] & (\vec E, \vec B) \arrow[uu, "q"', "\text{Lorentz}"] \arrow[r, "d"] \arrow[d, "\varepsilon"', shift right] \arrow[dr, "\sigma"] & (\vec 0, 0) & \text{constitutive} \\
\text{source eq's:} & & (\vec C, U) \arrow[r, "d"] & (\vec D, \vec H) \arrow[u, "\mu"', shift right] \arrow[r, "d", "\text{Maxwell}"'] & (\rho, \vec J) \arrow[r, "d", "\text{continuity}"'] & 0
\end{tikzcd}$$

gauge potentials fields current

* Motion in a magnetic field:

$$m\dot{\vec{v}} = \dot{\vec{p}} = \vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \qquad \text{1st order in } \vec{v} \text{ if } \vec{E} = \vec{0}\,!$$

let $\hat{z}$ be oriented along $\vec{B}$ so $\vec{B} \equiv \hat{z} B$, $\eta = v_x + i v_y \equiv \dot{\xi}$

$$m\dot{\eta} = \dot{\vec{p}} = \vec{F} = q\vec{v} \times \vec{B} = -qB\,\hat{z} \times \vec{v} = -qBi\eta$$

$$\dot{\eta} = -i\omega\eta \quad \text{where} \quad \omega \equiv \frac{qB}{m}$$ [cyclotron frequency]

$$d\ln\eta = \frac{d\eta}{\eta} = -i\omega dt$$

$$\ln \eta/\eta_0 = -i\omega t \qquad \eta = \eta_0 e^{-i\omega t}$$

$$\xi = x + iy = \xi_0 + \int_0^t \eta\, dt$$

$$= \xi_0 + i\omega\eta_0(1 - e^{-i\omega t})$$