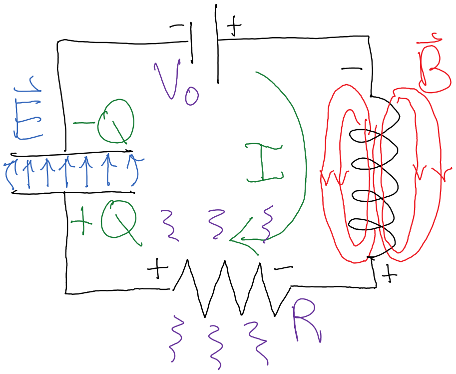
L18 Impedance analogy

Saturday, September 26, 2020 08:10

LRC-circuit



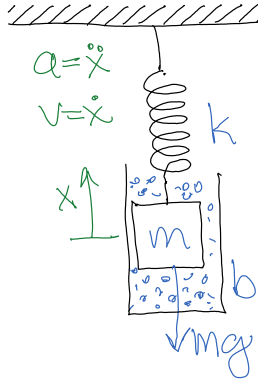
$V_L = L\dot{I}$

$V_R = RI$

$V_C = \frac{1}{C}Q$

$V_0 = \mathcal{E}$

Damped Oscillator



$F_\Sigma = ma$

$F_k = -kx$

$F_b = -bv$

$F_g = mg$

$$L\underbrace{\ddot{Q}_0}_{\ddot{Q}} + R\underbrace{\dot{Q}_0}_{\dot{Q}} + \frac{1}{C}(\underbrace{Q_0 + C}_{Q}\mathcal{E}) = 0 \quad \mathcal{E}(t)$$

$$m\underbrace{\ddot{x}_0}_{\ddot{x}} + b\underbrace{\dot{x}_0}_{\dot{x}} + k(\underbrace{x_0 - \tfrac{mg}{k}}_{x}) = 0 \quad F_0(t)$$

$\beta = R/2L \quad \omega_0^2 = 1/LC \quad f_0 = \mathcal{E}/L$

$\beta = b/2m \quad \omega_0^2 = k/m \quad f_0 = F_0/m$

Universal: $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \cancel{f_0}\ 0$ OR $\boxed{Dx = f_0}$ $D = \frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2$ (linear operator / matrix)

inhomogenous: $f_0 \neq 0$

homogenous: $f_0 = 0$

let $x = e^{st}$, $(s^2 + 2\beta s + \omega_0^2)x = 0$

$$= (s+\beta)^2 - (\beta^2 - \omega_0^2) = (s+\beta-\beta_1)(s+\beta+\beta_1)$$

$s = -\beta \pm \beta_1 = -\beta \pm i\omega_1$ $\quad \beta^2 - \omega_0^2 = \beta_1^2 = -(\omega_1^2 = \omega_0^2 - \beta^2)$

overdamped: β_1 real $(\omega_0 < \beta)$

$$x = C_1 e^{(-\beta+\beta_1)t} + C_2 e^{(-\beta-\beta_1)t} = e^{-\beta t}\Big(B_1 \underbrace{\cosh(\beta_1 t)}_{\lim_{\beta_1\to 0} = 1} + B_2 \underbrace{\sinh(\beta_1 t)}_{\lim_{\beta_1\to 0} = \beta_1 t}\Big)$$

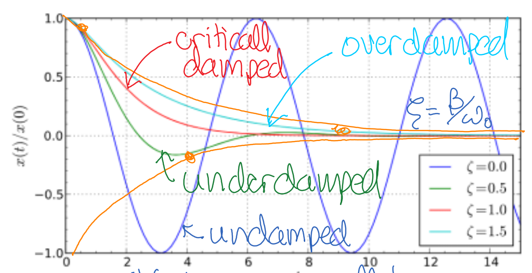
underdamped: ω_1 real $(\beta < \omega_0)$

$$x = C_1 e^{(-\beta+i\omega_1)t} + C_2 e^{(-\beta-i\omega_1)t} = e^{-\beta t}\big(B_1 \overbrace{\cos(\omega_1 t)} + B_2 \overbrace{\sin(\omega_1 t)}\big)$$

critically damped: $\omega_1 = \beta_1 = 0$ $(\beta = \omega_1)$

[absorb β_1 or ω_1 into C_2]

$$x = B_1 e^{-\beta t} + B_2 t e^{-\beta t}$$



wiki: harmonic oscillator

Initial conditions:

$$x_0 = B_1$$
$$\dot{x}_0 = -\beta B_1 + \underset{-i\beta_1}{\omega_1} B_2$$
$$\omega_1 B_2 = \dot{x}_0 + \beta x_0$$

$$x = e^{-\beta t}\left(x_0 \cos\omega_1 t + \frac{\dot{x}_0+\beta x_0}{\omega_1}\sin\omega_1 t\right) \quad \text{[underdamped]}$$
$$= e^{-\beta t}\left(x_0 \cosh\beta_1 t + \frac{\dot{x}_0+\beta x_0}{\beta_1}\sinh\beta_1 t\right) \quad \text{[overdamped]}$$
$$= e^{-\beta t}\left(x_0 + (\dot{x}_0+\beta x_0)t\right) \quad \text{[critically damped]}$$

Impedance Analogy:

$$I = I_0 e^{i\omega t} \rightarrow \dot{I} = i\omega I \qquad Q = \frac{1}{i\omega} I$$

$$V = RI \rightarrow (Z = R + iX)\, I \qquad F = bv = Zv$$

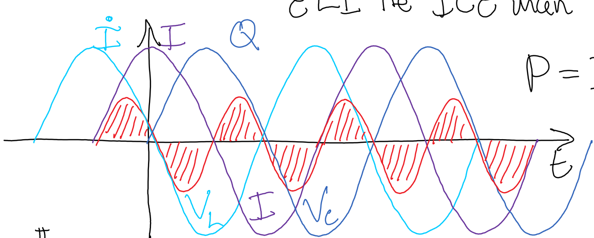
$$\overbrace{\Big[\underbrace{R}_{\text{resistance}} + \underbrace{\left(i\omega L + \frac{1}{i\omega C}\right)}_{\text{reactance}}\Big]}^{\text{impedance}} I \qquad = \left[b + i\left(\omega m - k/\omega\right)\right] v$$

$$X = \omega L - \frac{1}{\omega C}$$

also treat ma as "force"

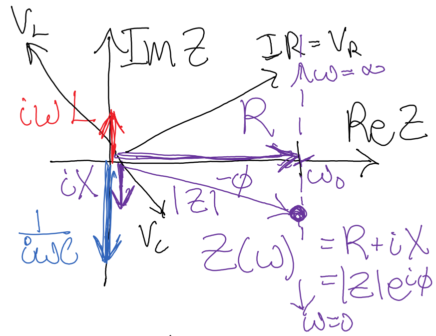
ELI the ICE man

$$P = IV$$



$$= \int_0^{\pi} P\,dt / 2\pi$$

$$\langle P \rangle = \tfrac{1}{2} V I^* = \tfrac{1}{2} Z I^* I = \tfrac{1}{2}(R + iX) I^2$$



R: dissipative – heat loss

X: reactive – transferred.