L20 Coupled oscillators

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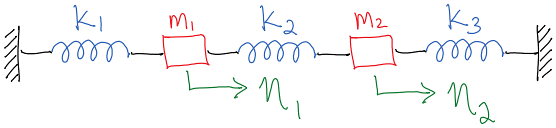
Problem: system of oscillating particles (for example: masses on springs) [Taylor 11.1-3]
Our goal is not only to solve this problem, but to gain insight into the linear algebra.

$$T = \frac{1}{2} m_1 \dot{\eta}_1^2 + \frac{1}{2} m_2 \dot{\eta}_2^2 = \frac{1}{2} \dot{\vec{\eta}}^T M \dot{\vec{\eta}}$$



$$V = \frac{1}{2} k_1 \eta_1^2 + \frac{1}{2} k_2 (\eta_2 - \eta_1)^2 + \frac{1}{2} k_3 \eta_2^2 = \frac{1}{2} \vec{\eta}^T K \vec{\eta}$$

(T, V matrices in Goldstein)

$$\mathcal{L} = \frac{1}{2} \dot{\vec{\eta}}^T M \dot{\vec{\eta}} - \frac{1}{2} \vec{\eta}^T K \vec{\eta}, \quad \vec{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \quad M = \begin{pmatrix} m_1 & \\ & m_2 \end{pmatrix} \quad K = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{pmatrix}$$

(from the general expansion about a local minimum in V)

$$V = \overset{\text{gauge invariance}}{V_0} + \underbrace{\nabla_{\vec{q}} V}_{\text{"gradient"}} \Big|_{\vec{q}_0}^{0 \text{ local minimum}} \cdot \underbrace{(\vec{q} - \vec{q}_0)}_{\vec{\eta}} + \frac{1}{2} (\vec{q} - \vec{q}_0)^T \underbrace{(\nabla_{\vec{q}} \nabla_{\vec{q}}^T V)}_{\text{"Hessian" } K} \Big|_{\vec{q}_0} (\vec{q} - \vec{q}_0)$$

$$M \vec{a} \omega^2 + \cancel{\underbrace{B \vec{a} i\omega}} + K \vec{a} = 0 \qquad \left(-\frac{d}{dt} \frac{\partial}{\partial \dot{\vec{\eta}}} + \frac{\partial}{\partial \vec{\eta}} \right) (T - V) = 0 \qquad K_{ij} = \frac{\partial^2 V}{\partial \eta_i \partial \eta_j}$$

Lagrange's equations relate $\mathcal{L} = T - V$ to the corresponding $\vec{F} = m\vec{a}$ equations (one per particle)

$$\vec{p} = \partial_{\dot{\vec{\eta}}} T = M \dot{\vec{\eta}} \qquad \begin{pmatrix} m_1 \ddot{\eta}_1 \\ m_2 \ddot{\eta}_2 \end{pmatrix} = M \ddot{\vec{\eta}} = \dot{\vec{p}} = m\vec{a} = \vec{F} = -\partial_{\vec{\eta}} V = -K \vec{\eta} = -\begin{pmatrix} (k_1 + k_2)\eta_1 + -k_2 \eta_2 \\ -k_2 \eta_1 + (k_2 + k_3) \eta_2 \end{pmatrix}$$

Form a system of constant linear equations by factoring out the temporal eigenfunctions:

$$-K \underbrace{\vec{a}}_{A = (\vec{a}_1 \vec{a}_2 \ldots)} = M \underbrace{\vec{a}}_{A} \underbrace{\omega^2}_{D = W^2} \qquad \vec{\eta} = \vec{a} e^{i\omega t} \qquad \dot{\vec{\eta}} = i\omega \vec{\eta} \qquad \ddot{\vec{\eta}} = -\omega^2 \vec{\eta}$$

This looks like an eigenvalue equation, except with two matrices!
Thus we need simultaneous diagonalization. Often M is diagonal (as in our case)

If we try to make a regular eigensystem $(M^{-1} K) A = A W^2$ the operator isn't Hermitian.

However it is self-adjoint under the metric M: $\widetilde{M^{-1} K} = M^{-1} (M^{-1} K)^T M = M^{-1} K$

It is generalized in that sense. You can directly solve the generalized characteristic equation

$$|K - M\omega^2| = 0 \text{ for } \omega_i^2 \text{ } (\lambda_i \text{ in Goldstein}) \text{ then } (K - M\omega_i^2) \vec{a}_i = 0 \text{ for } A = (\vec{a}_1, \vec{a}_2, \ldots)$$

normalized such that $A^T M A = I$ since $\omega_i^{*2} (\vec{a}_i^T M \vec{a}_j) = \vec{a}_i^T K \vec{a}_j = (\vec{a}_i^T M \vec{a}_j) \omega_j^2$

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(before and after) because $M^T = M$, $K^T = K$ and therefore, $(\omega_i^* - \omega_j)(\vec{a}_i^T M \vec{a}_j) = 0$

Finally, $KA = MAW^2$ where $W = \begin{pmatrix}\omega_1 & & \\ & \omega_2 & \\ & & \ddots\end{pmatrix}$ and $A^T K A = A^T M A W^2 = W^2$

As in QM, the spectral decomposition is $K = MAW^2A^{-1} = MAW^2A^TM$

In terms of normal coordinates [components!] $\vec{\xi}$ in the eigenbasis $\vec{\eta} = A\vec{\xi}$

the Lagrangian is $\mathcal{L} = T - V = \frac{1}{2}\dot{\vec{\eta}}^T M \dot{\vec{\eta}} - \frac{1}{2}\vec{\eta}^T K \vec{\eta} = \frac{1}{2}\dot{\vec{\xi}}^T\dot{\vec{\xi}} - \frac{1}{2}\vec{\xi}^T W^2 \vec{\xi}$

with the solution $\ddot{\vec{\xi}} + W^2\vec{\xi} = \vec{0} \Rightarrow \vec{\xi}(t) = \mathrm{Re}\{e^{iWt}(\vec{\xi}_0 + \frac{i}{W}\dot{\vec{\xi}}_0)\} = \frac{1}{2}(\{\dots\} + \{\dots\}^*)$

In original components, $\vec{\eta}(t) = A\vec{\xi}(t) = \mathrm{Re}\{Ae^{iWt}(A^TM\vec{\eta}_0 + iW^{-1}A^TM\dot{\vec{\eta}}_0)\}$

(and original matrices) $= \mathrm{Re}\{e^{i\sqrt{\cancel{M}K}t}(\vec{\eta}_0 + i\sqrt{\cancel{K}M}\,\dot{\vec{\eta}}_0)\}$ $\tilde{A} = A^TM$

Same form as the QM! $|\psi(t)\rangle = U|\psi_0\rangle$ where $HU = i\hbar\dot{U} \Rightarrow U = e^{-iHt/\hbar}$

An alternative analysis is to transform $M \to M' = I$ with $\vec{a}' = B\vec{a}$ and then solve the orthonormal eigenvalue problem $KC = CW^2$. The final solution $A = CB$ is the same.

The use of $(\vec{\xi}_0 + \frac{i}{W}\dot{\vec{\xi}}_0)$ for initial conditions is a shortcut for the full solution involving the most general solution involving both $\pm\omega_i$. Since $\dot{\vec{\eta}} = i\omega\vec{\eta}$, dividing by $i\omega$ allows us to encode the both initial conditions as the real and imaginary parts a single complex vector. Analyzing just one frequency component,

$$\vec{\xi}_0(t) = C_+e^{i\omega t} + C_-e^{-i\omega t} \qquad \vec{\xi}_0 = C_+ + C_-e^{-i\omega t} \qquad 2C_+ = \vec{\xi}_0 + \frac{i}{\omega}\dot{\vec{\xi}}_0$$

$$\dot{\vec{\xi}}_0(t) = i\omega C_+e^{i\omega t} - i\omega C_-e^{-i\omega t} \qquad \dot{\vec{\xi}}_0 = i\omega C_+ - i\omega C_-e^{-i\omega t} \qquad 2C_- = \vec{\xi}_0 - \frac{i}{\omega}\dot{\vec{\xi}}_0 = 2C_+^*$$

Adding the $-\omega$ component (the complex conjugate) is equivalent to taking the real part.

Note that while the damped harmonic oscillator has a closed form solution, a system of damped oscillators has three matrices K, B, M. As before, we can treat the mass matrix M as a metric, but the stiffness K and damping B matrices must commute in order to have an analytic solution.

This simultaneously involves both types of simultaneous diagonalization!

For nonlinear potentials, the harmonic approximation is only valid for small amplitudes.

The double pendulum involves 'damping' terms arising from crossed terms in the metric. Thus periodic solutions only exist for small amplitudes where $\dot{\vec{\eta}}$ can be neglected.

Background notes:

1. *Coordinate transformations on operators:* let $\vec{w} = Q\vec{v}$ and $\vec{v}' = R\vec{v}$

then $\underbrace{R\,\vec{w}} = \underbrace{R\,Q\,(R^{-1}}\underbrace{R)\,\vec{v}}$ or $\vec{w}' = Q'\vec{v}'$ where $Q' = RQR^{-1}$

in components, $Q^{i'}{}_{j'} = R^{i'}{}_{i}\,Q^{i}{}_{j}\,R^{j}{}_{j'}$ where $R^{j}{}_{j'} \equiv R^{-1\,j'}{}_{j}$ "similarity transform"

2. *Coordinate transforms on the metric:* let $d = \vec{w}^T G \vec{v}$ and $\vec{v}' = R\vec{v}$

then $d = \vec{w}^T (R^{-1}R) G (R^{-1}R)\vec{v} = (R\vec{w})^T (R^{-1T} G R^{-1})(R\vec{v}) = \vec{w}'^T G' \vec{v}'$

where $G' = R^{-1T} G R^{-1}$ $\qquad g_{i'j'} = R^{i}{}_{i'}\, g_{ij}\, R^{j}{}_{j'}$ "congruence transform"

The difference is that operators are (1,1) tensors while the metric is a (0,2) tensor.
The operator equivalent of the metric (by raising one index is just the identity!

$$g^{i}{}_{k} = g^{ij} g_{jk} = \delta^{i}{}_{k} \qquad g^{i'}{}_{k'} = R^{i'}{}_{i}\,\delta^{i}{}_{j}\,R^{j}{}_{j'} = R^{i'}{}_{j} R^{j}{}_{j'} = \delta^{i'}{}_{j'}$$

3. *Definition of adjoint:* a) *Vector* $\vec{v}$: $\quad \tilde{v} \equiv v\bullet = v^T G \qquad \tilde{v}_i = v^{\bar{i}} g_{\bar{i}i}$

b) *Operator* Q : $\quad \tilde{v}\,\tilde{Q} \equiv \widetilde{Qv}$ or $v^T G \tilde{Q} = (Qv)^T G = v^T Q^T G$

thus $\tilde{Q} = G^{-1} Q^T G$

components: $\tilde{Q}^{i}{}_{j} = g^{i\bar{i}}\, {Q^T}_{\bar{i}}{}^{\bar{j}}\, g_{\bar{j}j} = g^{i\bar{i}}\, Q^{\bar{j}}{}_{\bar{i}}\, g_{\bar{j}j} = Q_{j}{}^{i}$

thus G raises/lowers indices ($\updownarrow$): $g_{i\bar{i}}\, Q^{\bar{i}}{}_{j} = Q_{ij}$

the transpose ^{T} switches ($\leftrightarrow$): ${Q^T}_{\bar{j}}{}^{\bar{i}} = Q^{\bar{i}}{}_{\bar{j}}$

the adoint $\sim$ does both ($\nwarrow\!\searrow$): to keep $\tilde{Q}$ an operator