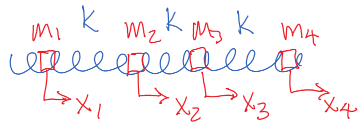
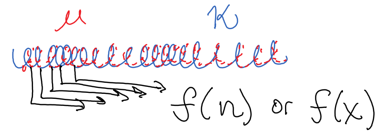
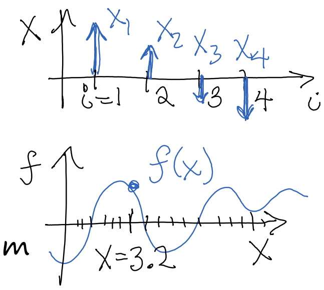
L22 Waves and Dispersion

Monday, October 28, 2019 08:27

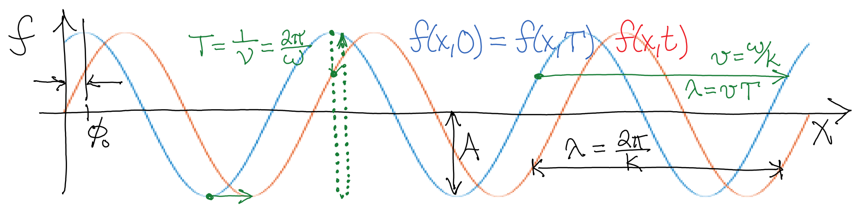
- What is a wave?







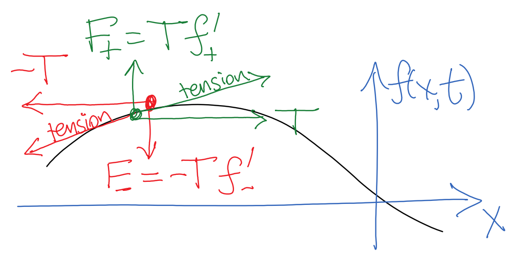
1) Mode of oscillation in system of infinite continuous coupled degrees of freedom
   Wave function: $f(x,t)$ state is indexed by continous x vs discrete i: $x_i \to f(x)$
   Wave equation: $\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) f(x,t) = \rho$ equation of motion (NII) is 2nd order PDE vs ODE $\quad M\ddot{\vec{x}} + \mathrm{K}\vec{x} = \vec{0}$
   Solution: any function $f(x,t) = g(u)$ where $u = x \mp vt$, satisfies the wave equation with velocity $\pm v$
   Example: pure frequency waves $f(x,t) = e^{i(kx-\omega t)}$ where $g(u) = e^{iku}$ velocity $v = \frac{\omega}{k}$
   $f(x) = \sum_k A_k e^{i(kx-\omega t)}$ is superposition of collective "modes" $(\omega, \vec{k})$ vs "particles" $(t, \vec{x})$.



$f(x,t) = A\cos(kx - \omega t - \phi_0)$
wave fn $f(x,t)$ (position)
ampl. dist. $A(k,\omega)$ (frequency)

2) Transfer of energy and momentum (NIII) without accompanying material
   Medium: point-like properties k, b, m replaced by material properties $T, \mu \to v, Z$
   Dispersion relation $\omega(k) \to v$ propagation, and impedance $Z \to P$ reflection/transmission.
   Linear medium (wave eq.): superposition/interference; polarization

   Impedance: the transfer energy/momentum.
   - By NIII, the forces between two links on a string are equal and opposite on each link.
   - The balanced horizontal tension T keeps x fixed
   - $F_\pm$ transfers momentum and energy along string



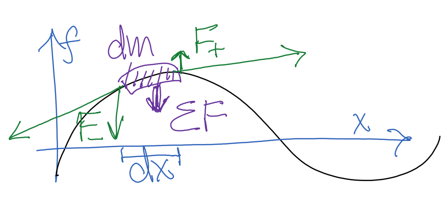
$$d\vec{p} = d(m\vec{v}) = m\vec{a}\,dt = \vec{F}dt \qquad \text{"momentum flux"} \qquad \frac{d\vec{p}}{dt} = F_\pm = \pm T f'$$

$$dE = d(\tfrac{1}{2}mv^2) = mv\,dv = m\vec{a}\cdot\vec{v}dt = \vec{F}\cdot\vec{v}\,dt = P dt \qquad \text{"power"} \qquad \frac{dE}{dt} = P = F\dot{f}$$

- Derivation of the wave equation
  - By NII, consider net forces on an element of string

$$\Sigma\vec{F} = F_+ + F_- = T(f'_+ - f'_-) = \mu dx\,\ddot{f} = dm\,\vec{a}$$

$$\frac{dTf'}{dx} = \mu\ddot{f} \quad \text{or} \quad (\partial_x T \partial_x - \mu\partial_t^2) f(x,t) = 0$$



  - this is the "wave equation", (equation of motion) which evolves the "wave function" (state)

- *General solution of the wave equation: show that $f(x,t) = g(x \mp vt)$ satisfies the wave eq.*

Let $g(u)$ be any function with derivatives $g'(u)$
subst. $u(x,t) = x \mp vt$, so $f(x,t) = g(x \mp vt)$

$$\partial_x f(x,t) = g'(u)\frac{\partial u}{\partial x} = g'(u) \quad \text{and} \quad \partial_t f(x,t) = g'(u)\cdot\frac{\partial u}{\partial t} = \mp v\, g'(u)$$
$$\partial_x^2 f(x,t) = g''(u)\frac{\partial u}{\partial x} = g''(u) \quad \text{and} \quad \partial_t^2 f(x,t) = \mp v\, g''(u)\frac{\partial u}{\partial t} = v^2 g''(u)$$

$$(\partial_x T \partial_x - \mu\, \partial_t^2) f = (T - \mu v^2)\, g''(u) = 0 \quad \text{if} \quad v = \sqrt{T/\mu}$$

the velocity is a property of the medium, not the mode!

Note $g(x-a)$ shifts the wave to the right by "a".
thus $g(x \mp vt)$ travels at velocity $\pm v$.

- *Eigenfunctions of the wave equation operators (continuous analog of Matrix operators)*

$$\underbrace{\partial_x}_{\text{operator}} e^{ikx} = \underbrace{ik}_{\text{eigenvalue}}\, e^{ikx} \qquad \underbrace{\partial_t}_{\text{operator}} \underbrace{e^{-i\omega t}}_{\text{eigenfunction}} = \underbrace{-i\omega}_{\text{eigenvalue}}\, e^{-i\omega t} \qquad \underbrace{M}_{\substack{\text{matrix}\\ \text{operator}}} \underbrace{\vec{v}}_{\substack{\text{eigen}\\ \text{vector}}} = \underbrace{\lambda}_{\substack{\text{eigen}\\ \text{value}}} \vec{v}$$

The PDE "wave function" is turned into an algebraic "dispersion relation"
by applying the wave equation to the product eigenfunctions (separation of variables)

$$(\partial_x T \partial_x - \mu\, \partial_t^2)\, e^{i(kx-\omega t)} = \left((ik)T(ik) - \mu(-i\omega)^2\right) e^{i(kx-\omega t)} = 0$$

$$Tk^2 - \mu\omega^2 = 0 \quad \text{or} \quad \omega = \sqrt{T/\mu}\, k = v\, k$$

The general solution can be written as a linear combination of "basis eigenfunctions"

$$f(x,t) = \sum\!\!\!\!\!\!\int_k \underbrace{A_k}_{\text{component}}\, \underbrace{e^{i(kx-\omega t)}}_{\text{basis.}} \qquad \text{(Fourier transform)}$$

The dispersion relation determines the velocity of a pure frequency wave component.

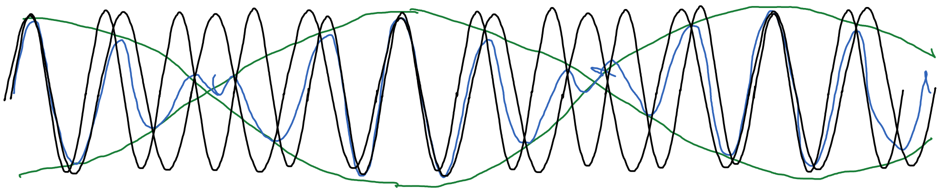
$$e^{i(kx-\omega t)} = e^{ik(x-\frac{\omega}{k}t)} = g(x - v t) \qquad \text{where} \quad g(u) = e^{iku}$$

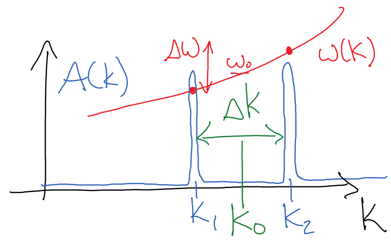
$$\text{thus} \quad v = \frac{\omega}{k} = \frac{2\pi}{k}\cdot\frac{\omega}{2\pi} = \frac{\lambda}{T} = \lambda\nu$$

λ = wavelength
$\frac{1}{\nu} = T$ = period.

If the material properties are functions of the spatial frequency k,
then different frequencies will have different velocities $v = \sqrt{T(k)/\mu(k)} = \omega(k)/k$,
and a multi-frequency wave packet will spread out over time (dispersion).
For this reason, the function $\omega(k) = k\, v(k)$ is called the dispersion relation.

- *The simplest system to analyze dispersion is the interference two pure-frequency waves.*
  - *Two instruments slightly out of tune produce a beating tone due to alternating (+) and (−) interference of the two waves (either in space or time)*
  - *Maximum destructive interferences occurs when both waves have the same amplitude.*
  - *The corresponding frequency spectrum $A(k)$ has two identical peaks at k_1 and k_2.*
  - *The spectrum $A(k)$ can be plotted on the same k-axis as the dispersion relation $\omega(k)$.*





$$f(x,t) = A e^{i(k_1 x - \omega_1 t)} + A e^{i(k_2 x - \omega_2 t)} = A\left(e^{i\phi_1} + e^{i\phi_2}\right) \qquad \phi_i \equiv k_i x - \omega_i t \quad i = 1,2$$

$$= A e^{i\bar{\phi}}\left(e^{i\Delta\phi} + e^{-i\Delta\phi}\right) = A e^{i\bar{\phi}} \cdot 2\cos\Delta\phi \qquad \phi_{1,2} = \bar{\phi} \pm \Delta\phi \qquad \bar{\phi} = \tfrac{1}{2}(\phi_1 + \phi_2), \quad \Delta\phi = \tfrac{1}{2}(\phi_2 - \phi_1)$$

$$= A \underbrace{e^{i(\bar{k}x - \bar{\omega}t)}}_{\text{phase/carrier wave}} \cdot \underbrace{2\cos(\Delta k \cdot x - \Delta\omega\, t)}_{\text{group/packet/beats/modulation.}} \qquad \cos\Delta\phi = \tfrac{1}{2}\left(e^{i\Delta\phi} - e^{-i\Delta\phi}\right)$$

- the "carrier wave" (average frequency) travels at the average "phase velocity" of the two waves $\quad v_\phi = \bar{\omega}/\bar{k} \rightarrow \dfrac{\omega}{k}$

- the "envelope/modulation packet" travels at the "group velocity" (slope of dispersion relation) $\quad v_g = \Delta\omega/\Delta k \rightarrow d\omega/dk$

- the envelope gets wider as the beating frequencies get closer
  $\Delta k \cdot \Delta x \approx \pi \quad \Rightarrow \quad \Delta x \approx \pi/\Delta k$

- *Summary: a nonlinear dispersion relations cause a wave packet to spread out*
  Beats: $f(x,t) = e^{i\phi_1} + e^{i\phi_2} = 2\cos\hat{\phi}\, e^{i\bar{\phi}}$ *where* $\phi = kx - \omega$ *and* $\phi_{1,2} = \bar{\phi} \pm \hat{\phi}$
  Carrier wave $\bar{\phi} = \frac{\phi_1 + \phi_2}{2}$ *travels at the phase velocity* $v_\phi = \frac{\bar{\omega}}{\bar{k}} \approx \frac{\omega}{k}$,
  Wave packet $\hat{\phi} = \frac{\phi_2 - \phi_1}{2}$ *travels at the group velocity* $v_g = \frac{\hat{\omega}}{\hat{k}} \approx \frac{d\omega}{dk}$

- *The same principle applies to general "wave packets" with a localized spectrum:*
  - *The "phase velocity"* $v_\phi = \omega(\bar{k})/k$ *describes the velocity of the carrier wave ripples.*
  - *The "group velocity"* $v_g = d\omega(\bar{k})/dk$ *describes the velocity of the packet envelope*
  - *Both velocities are evaluated at the average wavelength* $\bar{k}$ *of the spectrum*