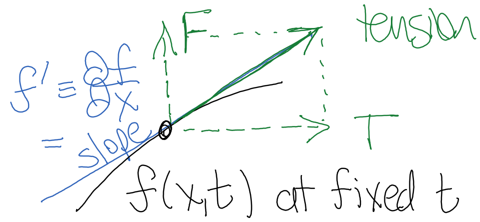
L23 Reflections: Characteristic impedance

Saturday, October 10, 2020 09:37

Overview: L22 treated waves in context of a string with distributed mass μ under tension directed ALONG the string with FIXED horizontal component T.
By similar triangles the vertical force from tension is $F = Tf'$.



$F = Tf'$

L22 described 1) waves as modes of oscillation/propagation
L23 will discuss 2) waves as transfer of momentum/energy

1) Summary of wave as "mode of oscillation" in a medium:

$f(x,t)$ "wave function" describes the "trajectory" of the "state" $f(x)\big|_t$

N II: $\underbrace{\sum F = ma}_{\text{applied to } dx} \Rightarrow \underbrace{(Tf')' = \mu \ddot{f}}_{\substack{\text{wave equation} \\ \text{(equation of motion)}}}$ or $\underbrace{\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)}_{\text{d'Alembertian}} \underbrace{f(x,t)}_{\substack{g(x \mp vt) \\ \text{general solution}}} = 0$ or $\underbrace{\frac{1}{v^2}\omega^2 - k^2 = 0}_{\substack{\text{dispersion relation} \\ (\text{eiganvector } e^{ikx} \cdot e^{-i\omega t})}}$

Wave characteristic property of the medium (string)
"velocity" of propagation (evolution) of the wave.

$v = \omega/k = \sqrt{T/\mu}$

2) Waves as the "transfer of momentum/energy" without the accompanying transfer of material:
Conservation of momentum/energy: transfer and attenuation across boundaries

N III $F_+ = -F_- \Rightarrow$ boundary conditions: $\underbrace{\Delta f = 0}_{\text{(continuity)}}$ & $\Delta(Tf') = \Delta(\pm Z\dot{f}) = Z_0 \dot{f}$

Property of the medium: "characteristic impedance"
determines relative amplitude, not velocity of wave

$Z \equiv F/\dot{f} = \sqrt{T\mu}$

- Two material properties affect the wave propagation according to these two characteristics:

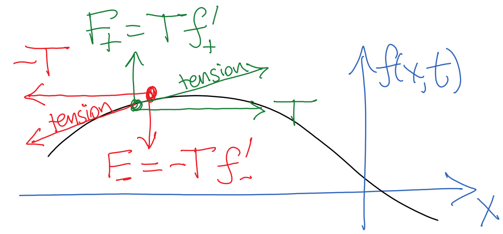
property	physical	→ wave
of medium	$\{T, \mu\}$	$\{v, Z\}$
of mode	$f(x,t)$	$A(\omega, k)$

	propagation dynamics		ref
bulk	$\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) f = 0$		L22
boundary	$\Delta f = 0$	$\Delta(\pm Z \dot{f}) = Z_0 \dot{f}$	L23

- Different types of waves are similar, but with different material properties and physics

medium	dim	L/T	pol	state	restoring force	inertia	dynamics	v	Z	ref:
• slinky	1	L(T)	1 (2)	f	κ	μ	$F = \kappa f'$	$\sqrt{\kappa/\mu}$	$\sqrt{\kappa\mu}$	L20/H06
• string	1	T	2	f	T	μ	$F = T f'$	$\sqrt{T/\mu}$	$\sqrt{T\mu}$	L22/H06
• gravity	2	L/T	1	η	g	ρ	$P = \rho g \eta$	$\sqrt{g/k}$	?	L24/H07
• surface	2	T	1	η	γ	λ, ρ	$P = \gamma \nabla_\perp^2 \eta$	$\sqrt{\frac{\gamma k}{\rho}}$	?	L25/H07
• elastic	3	L, T	3	$\vec{u}$	μ, λ	ρ	$\overleftrightarrow{\tau} = \lambda I \varepsilon_{kk} + 2\mu \overleftrightarrow{\varepsilon}$	$\sqrt{M/\rho}, \sqrt{G/\rho}$	?	L26/H07
• sound	3	L	1	P, ρ	γP	ρ	?	$\sqrt{\frac{\gamma P}{\rho}}$	?	L26/H07
• coaxial cable	1	T	1	q, I	$1/C$	L	$V = \frac{1}{C}Q = L\dot{I}$	$\frac{1}{\sqrt{LC}}$	$\sqrt{L/C}$	L18/bonus
• electromagnetic	3	T	2	$\vec{E}, \vec{B}$	$1/\varepsilon$	μ	$\nabla \times \vec{E} + \mu \frac{\partial \vec{H}}{\partial t} = 0$ $\nabla \times \vec{B} - \varepsilon \frac{\partial \vec{E}}{\partial t} = 0$	$\frac{c}{n} = \frac{1}{\sqrt{\mu\varepsilon}}$	$\sqrt{\mu/\varepsilon}$	PHY 417G
• quantum	3	N/A	2s+1	Ψ	?		$\left(\mathcal{H} = \frac{p^2}{2m} + V\right)\Psi$	$\frac{\hbar k}{2m}$	V?	PHY 520
• gravitation	3	Q	2	$g_{\mu\nu}$			$G_{\mu\nu} = \kappa T_{\mu\nu}$	c	?	PHY 605

- Transfer of energy and momentum
  - By NIII, the forces between two links on a string are equal and opposite on each link.
  - The balanced horizontal tension T keeps x fixed
  - $F_\pm$ transfers momentum and energy along string



$$d\vec{p} = d(m\vec{v}) = m\vec{a}\,dt = \vec{F}\,dt \qquad \text{"momentum flux"} \qquad \frac{dp}{dt} = F_\pm = \pm T f'$$

$$dE = d(\tfrac{1}{2}mv^2) = mv\,dv = m\vec{a}\cdot\vec{v}\,dt = \vec{F}\cdot\vec{v}\,dt = P\,dt \qquad \text{"power"} \qquad \frac{dE}{dt} = P = F\dot{f}$$

- Impedance relates the force to velocity, which relates momentum- (force) to energy-flux (power).

$$\vec{F} = m\vec{a} + b\vec{v} + k\vec{x} \equiv Z\vec{v} \quad \left(Z = i\omega m + b + \tfrac{1}{i\omega}k\right) \quad \text{for periodic motion}$$

- Equivalent, by the "Impedance analogy" with electrical circuits

$$V = L\dot{I} + IR + Q/C \equiv ZI \quad \left(Z = i\omega L + R + \tfrac{1}{i\omega C}\right) \quad \text{for tank circuit.}$$

$$\text{thus } F = Z\dot{f} \quad \text{and} \quad P = F\dot{f} = Z\dot{f}^2 \qquad \text{like } P = VI = I^2R$$

- Matched boundary conditions allows both energy and momentum transfer to down the string

$$\text{if } \Delta F = \Delta(Tf') = \Delta(Z\dot{f}) = 0 \quad \text{then momentum is conserved.}$$

$$\text{if } \Delta(F\dot{f}) = 0 \hookrightarrow \Delta\dot{f} = 0 \quad \text{then energy is also conserved.}$$

- Kinetic and potential energy density along the string:

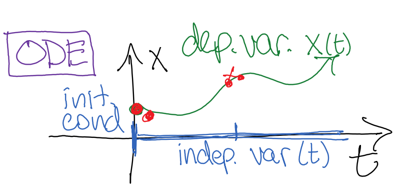
$$\mathcal{T} \equiv \frac{dT}{dx} = \boxed{\tfrac{1}{2}\mu\dot{f}^2} = -\tfrac{1}{2}\mu\omega^2 f \qquad \text{is the kinetic energy density (per length } dx \text{ of string)}$$

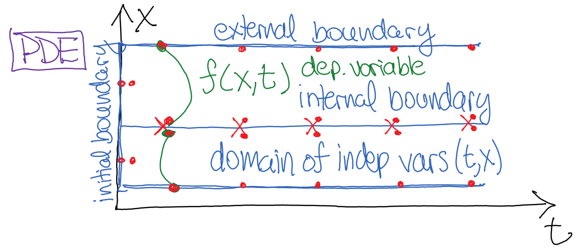
$$\delta\, dV \equiv -dF\,\delta f = -T\,df'\,\delta f' = \overset{\text{boundary term}}{d(-Tf'\delta f')} + Tf'\,d\delta f = Tf'\delta f'\,dx$$

$$\mathcal{V} \equiv \frac{dV}{dx} = \int_0^{f'} Tf'\delta f' = \boxed{\tfrac{1}{2}Tf'^2} = -\tfrac{1}{2}\underbrace{Tk^2}_{=\mu\omega^2} f = \mathcal{T} \qquad \text{potential energy density}$$

$$\underbrace{U \equiv \mathcal{T} + \mathcal{V} = \mu\dot{f}^2}_{\text{total energy density}} \qquad P = \underbrace{vU}_{\text{energy flux}} = \sqrt{T/\mu}\cdot\mu\dot{f}^2 = \underbrace{\sqrt{T\mu}}_{\text{impedance}}\,\dot{f}^2 = Z\dot{f}^2 = \underbrace{F\dot{f}}_{\text{power}}$$

- Boundary conditions (BCs)
  - We've been using BCs since day 1: an n^{th}-order ODE requires n initial conditions
    for example: NII $F = m\ddot{x}$ reqires $x_0, \dot{x}_0$, Hamilton's equations requires p_0, x_0 .
  - Independent variable ODE: (t) is linear. PDE: $(t, x, ..)$ is a whole region





  - External BCs: one per point on the boundary (like pins)

Elliptic:	$x^2 + y^2$	$\nabla^2 = \partial_x^2 + \partial_y^2 + ..$	1 boundary condition / side
Parabolic:	$t + x^2$	$\partial_t + \nabla^2$	1 initial condition + 1 boundary condition / side
Hyperbolic:	$t^2 - x^2$	$-\frac{1}{v^2}\partial_t^2 + \nabla^2$	2 initial conditions + 1 boundary condition / side

    While initial conditions determine the exact linear combination of basis solutions,
    Boundary conditions determine the ratio of amplitudes, and QUANTIZE the wavelength!

  - Internal (Continuity) BCs: 2 neighbouring BCs tied together (like shoelaces)
    They account for discontinuities in the domain (sudden change in string properties)
    Formed by integrating the n first-order differential equations across the boundary.

ODE $\circ \to \Delta$

$$\int_{-\varepsilon}^{\varepsilon} dt \begin{cases} \dot{x} = v \\ m\dot{v} = \underbrace{F}_{J\delta(t)} \end{cases} \Rightarrow \boxed{\begin{matrix} \Delta x = 0 \\ m\Delta v = J \end{matrix}}$$

$$x_2 = x_1 \big|_\partial \qquad \dot{x}_2 - \dot{x}_1 = J/m \big|_\partial$$

PDE $/ \to \Delta$

$$\int_{-\varepsilon}^{\varepsilon} dx \begin{cases} Tf' = F \\ F' = \underbrace{\mu}_{m\delta(x)} \ddot{f} \end{cases} \Rightarrow \boxed{\begin{matrix} \Delta f = 0 \\ \Delta F = m\ddot{f} \end{matrix}}$$

$$f_2 = f_1 \big|_\partial \qquad T_2 f_2' - T_1 f_1' = m\ddot{f} \big|_\partial$$

Vector PDEs $\quad \nabla \to \Delta\hat{n}$

$$\int_{-\varepsilon}^{\varepsilon} dn \left(\nabla = \hat{n}\frac{\partial}{\partial n} + \cancel{\hat{t}\frac{\partial}{\partial t}} + \cancel{\hat{s}\frac{\partial}{\partial s}}\right) \circ \ldots = \hat{n} \circ \int_{-\varepsilon}^{\varepsilon} d \ldots = \hat{n} \circ \Delta \ldots = \Delta\hat{n} \circ \ldots$$

example: $\nabla \cdot \vec{D} = \underbrace{\rho}_{\sigma\delta(n)} \Rightarrow \Delta D_n = \sigma \qquad \nabla \times \vec{H} = \underbrace{\vec{J}}_{\vec{K}\delta(n)} \Rightarrow \Delta H_t = K_s$

- Characteristic Impedance: The boundary condition $\Delta(F = Tf') = m\ddot{f}$ is not so useful because the wavelength λ and thus $k \sim i\,\partial_x$ can vary with position. However, the frequency $2\pi\nu = \omega \sim -i\,\partial_t$ is always the same; otherwise crests would have to pile up somewhere! Thus we will express the boundary conditions in terms of $\dot{f}$ instead of f' for pure frequency waves. This again allows us to convert the BCs from a differential to an algebraic equation:

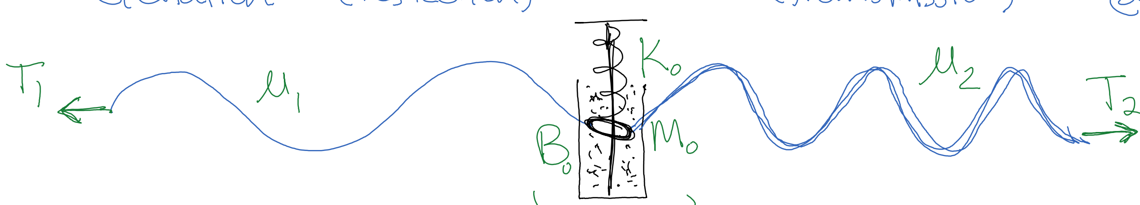
$$\Delta(F = Tf') = m\ddot{f}\Big|_\partial \quad\Rightarrow\quad \Delta(F = \pm Z\dot{f}) = Z_0\dot{f}\Big|_\partial$$

$$\text{where}\quad \underbrace{Z \equiv \mp F/\dot{f}}_{\text{definition}} = \mp T\underbrace{f'/\dot{f}}_{\text{using } f = e^{i(kx \mp \omega t)}} = \frac{\mp T\,ik}{\mp i\omega} = \underbrace{T/\sqrt{T/\mu}}_{\frac{\omega}{k} = v = \sqrt{T/\mu}} = \sqrt{T\mu}$$

The difference between characteristic impedance and normal impedance is that it represents power transmitted along the line, cumulative resistance along the line. Thus, a $50\,\Omega$ coaxial cable does NOT have $50\,\Omega$ if you short one end and measure the other! The power will be completed passed into a $50\,\Omega$ termination resistor, though, without reflection.

- EXAMPLE: reflection / transmission coefficients, two strings attached by a ring on a rod

$$f_1 = \underbrace{\overrightarrow{Ae^{i(k_1x - \omega t)}}}_{\substack{\text{forward}\\ \text{excitation}}} + \underbrace{\overleftarrow{Be^{i(-k_1x - \omega t)}}}_{\substack{\text{backward}\\ \text{(reflection)}}} \qquad f_2 = \underbrace{\overrightarrow{Fe^{i(k_2x - \omega t)}}}_{\substack{\text{forward}\\ \text{(transmission)}}} + \underbrace{\overleftarrow{\cancel{G}e^{i(-k_2x - \omega t)}}}_{\substack{\text{backwards}\\ \text{excitation}}}$$



$$Z_1 = \sqrt{T_1\mu_1} \qquad Z_0 = i\omega M_0 + B_0 + \tfrac{1}{i\omega}K_0 \qquad Z_2 = \sqrt{T_2\mu_2}$$

- The 2 external BCs fix A and $G = 0$ (amplitude of wave coming in from $-\infty$). Don't solve for A!
- The 2 internal BCs are used to solve for B and F as functions of A and G.
- The resulting values B/A and F/A are the "reflection" and "transmission" amplitudes, respectively, which are used to calculate the reflection R and transmission T coefficients (also absorption S by Z_0).