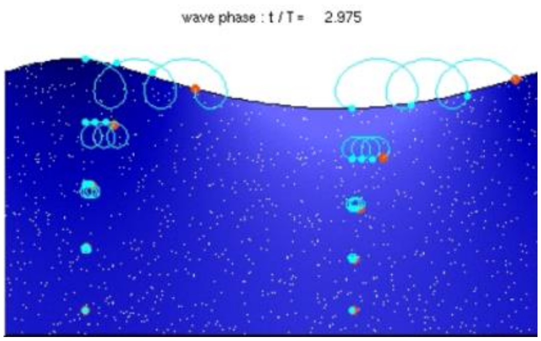
L24/25 Water waves

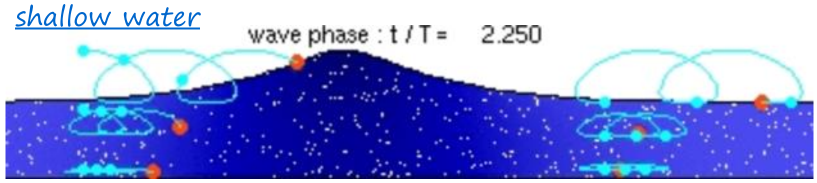
Saturday, October 10, 2020 09:36

Water waves:

deep water



shallow water



Wind waves (excitation) - not considered here
Airy theory (1st order) - sinusoidal oscillations
Stokes theory (2nd order) - sharp crests, Stokes drift
Cnoidal waves (nonlinear) - soliton solutions of the KdV eq.

3) Although the restoring force of gravity doesn't explicitly depend on position,
   but surface $\eta(x,t)$ between two fluids of different density adds the position dependence
   - Gravity tends to level out the water
   - **Bernoulli's law** relates gravity, pressure and velocity, via conservation of energy
2) Similar to a string, tension acts as a restoring force
   - The 2d version of tension is called **surface tension** (and the 3d version is pressure of Bernoulli's law)
1) Gravity waves involve circular motion in the plane,
   - Since water is incompressible to a good approximation, the water has to 'roll' out of the way
   - Thus while the surface is a 2d wave, water waves are really 3d waves!
   - Normal waves are also irrotational - no vortices, which allows application of **potential theory**
   - There is also a 'wave function' $\phi(x,z,t)$ in the bulk of the water, matched to the surface

1) Potential Theory

if $\underbrace{\nabla \times \vec{v} = 0}_{\text{"irrotational"}}$ then $\underbrace{\vec{v} = -\nabla \phi}_{\text{"conservative"}}$ $\underbrace{\phi(x,z)}_{\text{"potential"}} = -\int \vec{v} \cdot d\vec{l}$

Examples: if $\nabla \times \vec{F} = 0$ $\vec{F} = -\nabla V$ $V = -\int \vec{F} \cdot d\vec{x}$ potential energy

if $\nabla \times \vec{E} = 0$ $\vec{E} = -\nabla V$ $V = -\int \vec{E} \cdot d\vec{x}$ voltage

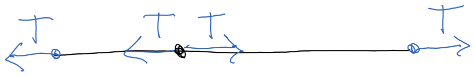
if $\underbrace{\nabla \cdot \vec{v} = 0}_{\text{"incompressible"}}$ then $\underbrace{\nabla^2 \phi = 0}_{\text{"harmonic"}}$ Laplace's equation

Solution by "separation of variables", same as wave eq'n

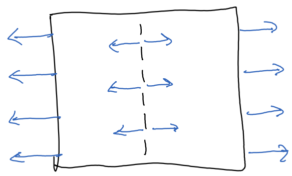
$$\nabla^2 \phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) e^{\alpha x} \cdot e^{\beta z} = (\alpha^2 + \beta^2)\, e^{\alpha x + \beta z}$$

let $\alpha = k$ (spatial freq of surface) then $\beta = ik$ so $\boxed{\phi = e^{ikx} e^{kz}}$

2) Surface tension – https://en.wikipedia.org/wiki/Surface_tension

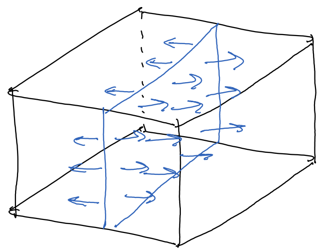
1d 

tension $T = F = \frac{dW}{dx}$ $dF = d(T f')$

2d 

$\gamma = F/l$

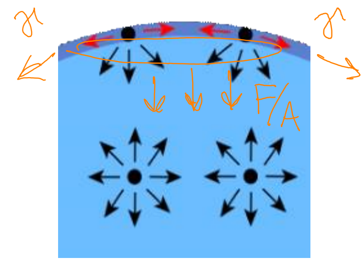
surface tension $\gamma = \frac{F}{l} = \frac{dW}{da}$ $\frac{F}{a} = \gamma \nabla_\perp^2 \underbrace{\eta(x,y)}_{\text{2d wave function}}$

3d 

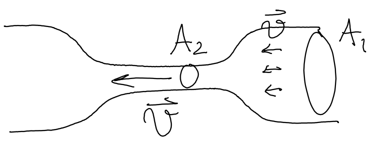
$P = F/A$

$\overleftrightarrow{\tau} = F/A$

pressure or stress $\begin{matrix} P \\ \overleftrightarrow{\tau} \end{matrix} = \frac{F}{A} = \frac{dW}{dV}$



3) Bernoulli's law



a) flux $= \vec{v} \cdot \vec{A} = \text{const} = \frac{\text{volume}}{\text{time}}$

b) $\underbrace{PAdx}_{dU} = \underbrace{Fdx}_{dW} = (m\frac{dv}{dt})dx = mdv \cdot v = \underbrace{d\tfrac{1}{2}mv^2}_{dT}$

add gravity: divide by V: $-\Delta P = \Delta \tfrac{1}{2}\rho v^2$

$$d\left(E = U_{therm} + U_{grav} + T = PV + mgh + \tfrac{1}{2}mv^2\right) = 0 \Rightarrow \boxed{P + \rho g h + \tfrac{1}{2}\rho v^2 = \text{const}}$$

and surface tension on surface $z = \eta(x,t)$ (const atm. pressure P)

$$P = -\rho g \eta + \gamma \nabla^2 \eta = \frac{\int F dx}{V} = \frac{\int ma}{V} \cdot d\vec{x} = \rho \int \frac{d\vec{v}}{dt} \cdot d\vec{x} = \rho \frac{d}{dt}\left(\int \vec{v} \cdot dx\right) = \rho\, \partial_t \phi$$

$$\boxed{-\partial_t \phi = -g\eta + \gamma/\rho\, \partial_x^2 \eta} \quad \text{where} \quad \boxed{-\nabla^2 \phi = 0}$$

this relates the surface (η) and bulk (ϕ) wave functions

and is the wave equation for gravity/surface tension waves.

Google images: cloud waves