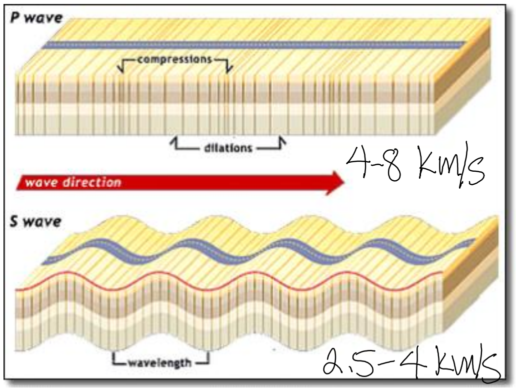
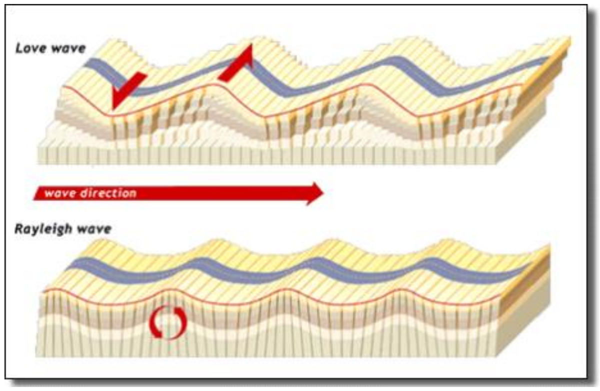
L26 Elastic waves

Friday, November 8, 2019 11:34

- Elastic (seismic) waves: Body (longitudinal and transverse) and Surface waves





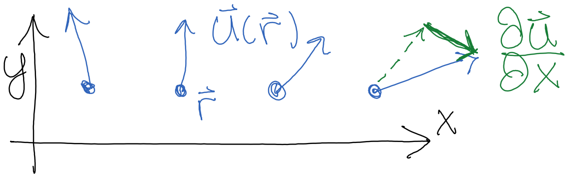
https://www.sms-tsunami-warning.com/pages/seismic-waves ; link to animations

- Other P-waves: L=1 orbital (QM); p-polarized light (E parallel to plane of incidence)
- Other S-waves: L=0 orbital (QM); s-polarized light (`senkrecht', German for perpendicular)

- Continuous analogs of Discrete variables in multi-dimensions

dimension	Discrete 0-d	------ Continuous 1-d (string)	2-d (surface)	3-d (bulk or body) --------	
index	i	x	x,y	$\vec{r}=(x,y,z)$	equilibrium pos.
displacement	$x_i(t)$	$f(x,t)$	$\eta(x,y,t)$	$\vec{u}(\vec{r},t)$	shift of particles
strain (stretch)	$x_{i+1}-x_i$	$f'(x,t)$	$\nabla_\perp \eta(x,y,t)$	$\tilde{\varepsilon}(\vec{r},t)$	$\varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i})$
stress (tension)	F	$F=F_\perp$	$\lvert\vec{F}\rvert=F_\perp/\ell$	$\tilde{\tau}(\vec{r})$	$\tau_{ij}=F_i/a_j$ (pressure, shear)
stiffness	k	$\kappa,\ T=F_\parallel$	$\gamma=F_\parallel/\ell$	λ,μ	$(E,G,K,M$ elastic moduli$)$
Hooke's law	$F=-k\Delta x$	$F=Tf'$	$\vec{\mathcal{F}}=\gamma\nabla_\perp\eta$	$\tilde{\tau}=\lambda \mathrm{Tr}\,\tilde{\varepsilon}\,\tilde{I}+2\mu\tilde{\varepsilon}$	$\tau_{ij}=\lambda\varepsilon_{kk}\delta_{ij}+2\mu\varepsilon_{ij}$
Newton's law	$F_i=m_i\ddot{x}_i$	$dF=\mu dx\,\ddot{f}$	$\nabla_\perp\cdot\vec{\mathcal{F}}=\sigma\ddot{\eta}$	$\nabla\cdot\tilde{\tau}=\rho\ddot{\vec{u}}$	$\tau_{ij,j}=\rho\ddot{u}_i$
wave eq.	$M\ddot{\vec{x}}=-K\vec{x}$	$\mu\ddot{f}=(Tf')'$	$\sigma\ddot{\eta}=\gamma\nabla_\perp^2\eta$	$\rho\ddot{\vec{u}}=M\nabla_\parallel^2\vec{u}+G\nabla_\perp^2\vec{u}$	

- Strain tensor (symmetric derivative of displacement field

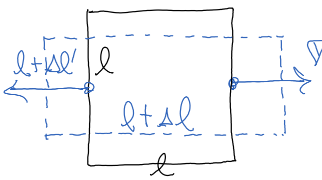


$$\overleftrightarrow{D} = \nabla \vec{u} = \begin{pmatrix} u_{x,x} & u_{y,x} & u_{z,x} \\ u_{x,y} & u_{y,y} & u_{z,y} \\ u_{x,z} & u_{y,z} & u_{z,z} \end{pmatrix}$$

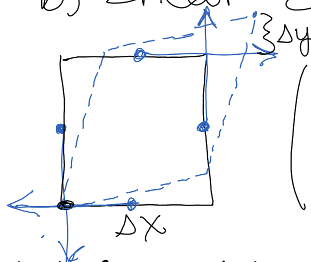
$$= \overleftrightarrow{\varepsilon} + \overleftrightarrow{r}$$

$\varepsilon_{ij} = \varepsilon_{ji}$ (symmetric strain tensor) $\quad r_{ij} = -r_{ji}$ (no stretch, just antisym rotation)

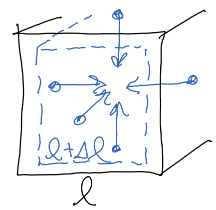
a) stretch $\varepsilon = \frac{\Delta l}{l}$ $\varepsilon' = \frac{\Delta l'}{l}$ b) shear $\varepsilon = \frac{\Delta y}{\Delta x}$ c) compression $\varepsilon = \frac{\Delta l}{l}$



$$\begin{pmatrix} \varepsilon & & \\ & \varepsilon' & \\ & & \varepsilon' \end{pmatrix}$$



$$\begin{pmatrix} & \varepsilon & \\ \varepsilon & & \\ & & \end{pmatrix}$$



$$\begin{pmatrix} \varepsilon & & \\ & \varepsilon & \\ & & \varepsilon \end{pmatrix}$$

- Stress tensor – force per area. Both the force and the area are vectors, so the stress tensor is a matrix

tensile stress $\sigma = T/A$ $\quad \begin{array}{c|c} & /\vec{A} \\ \hline \vec{F} & \begin{pmatrix} \sigma & & \\ & & \\ & & \end{pmatrix} \end{array}$ shear stress $\tau = F_x/A_y$ $\quad \begin{pmatrix} & \tau & \\ \tau & & \\ & & \end{pmatrix}$ pressure $p = F_{\parallel}/A$ $\quad \begin{pmatrix} -p & & \\ & -p & \\ & & -p \end{pmatrix}$

- stress tensor: $\overleftrightarrow{\tau}$ or $\overleftrightarrow{\sigma}$ is also symmetric $\sigma_{ij} = \sigma_{ji} = F_i/A_j$

$\vec{F}$ = force directed on the unit area $\vec{A}$

- similar to tension in a rope, stress is "intensive":

there are equal & opposite reaction pairs (NIII) that distribute the force across the medium (they don't add up).

- stress transfers momentum $\Delta\vec{p} = \overleftrightarrow{\sigma} \cdot d\vec{a}\, dt$ through the solid.

- Elastic moduli: 3d Hooke's law for elastic materials

$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$ relates each stress to every strain

σ_{ij} & ε_{kl} $\quad 3\times3 = 9$ elements $\Rightarrow$ 6 elements each, by symmetry.

$C_{ijkl} = C_{klij} = C_{jikl} = C_{ijlk}$ $\quad 3\times3\times3\times3 = 81 \Rightarrow 21$ elements by symmetry

For isotropic materials, rotational symmetry reduces C to 2 elements

"monopole" "dipole" "quadrupole"

$$\vec{r}_1 \vec{r}_2^{\,T} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} (x_2\ y_2\ z_2) = \begin{pmatrix} x_1x_2 & x_1y_2 & x_1z_2 \\ y_1x_2 & y_1y_2 & y_1z_2 \\ z_1x_2 & z_1y_2 & z_1z_2 \end{pmatrix}_{(9)} = \underbrace{\vec{r}_1 \cdot \vec{r}_2}_{\text{scalar (1)}} \overleftrightarrow{I} + \underbrace{\vec{r}_1 \times \vec{r}_2 \cdot \overleftrightarrow{M}}_{\text{vector (3)}} + \underbrace{\overleftrightarrow{Q}}_{\text{tensor (5)}}$$

Trace Antisymmetric Traceless symmetric

σ, ε symmetric

Schur's Lemma:

Only one scalar for each irreducible subspace:

Lamé parameters: λ, μ $\quad C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + 2\mu \delta_{ik} \delta_{jl}$

$$\boxed{\overleftrightarrow{\sigma} = \lambda \operatorname{Tr} \overleftrightarrow{\varepsilon} \cdot \overleftrightarrow{I} + 2\mu \overleftrightarrow{\varepsilon}} \qquad \text{components: } \sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$$

$$\overleftrightarrow{\varepsilon} = (\overleftrightarrow{\sigma} - \lambda \operatorname{Tr} \overleftrightarrow{\varepsilon} \cdot \overleftrightarrow{I})/2\mu \quad \text{where } \operatorname{Tr} \overleftrightarrow{\sigma} = (3\lambda + 2\mu) \operatorname{Tr} \overleftrightarrow{\varepsilon}$$

Elastic Moduli: physical representations of λ, μ

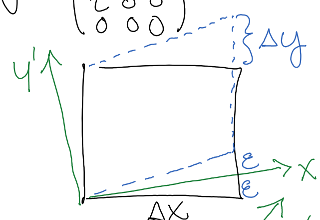
a) Young's Modus "E" : tensile stress $\overleftrightarrow{\sigma}_{ij} = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$\varepsilon = \varepsilon_{11} = \left(\sigma_{11} - \lambda \frac{\sigma_{11}}{3\lambda + 2\mu}\right)/2\mu = \sigma_{11} \frac{2(\lambda + \mu)}{2\mu(3\lambda + 2\mu)} \qquad E \equiv \frac{\sigma}{\varepsilon} = \boxed{\frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}}$$

$$\varepsilon' = \varepsilon_{22} = \left(\cancel{\sigma_{22}}^{\,0} - \lambda \frac{\sigma_{11}}{3\lambda + 2\mu}\right)/2\mu = \sigma_{11} \frac{-\lambda}{2\mu(3\lambda + 2\mu)} \qquad E' \equiv \frac{\sigma}{\varepsilon'} = \frac{-2\mu(3\lambda + 2\mu)}{\lambda}$$

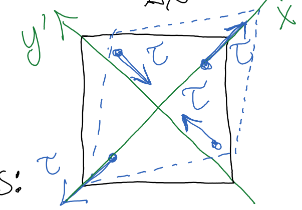
Poisson's Ratio "ν": $\qquad \nu \equiv \frac{-\varepsilon'}{\varepsilon} = \frac{E}{E'} = \boxed{\frac{\lambda}{2(\lambda + \mu)}}$

b) Shear Modulus "G" : shear stress $\overleftrightarrow{\sigma} = \begin{pmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$



Defined for $\overleftrightarrow{\sigma} = \begin{pmatrix} & \tau \\ & \end{pmatrix}$ $\quad \overleftrightarrow{D} = \underbrace{\begin{pmatrix} & \varepsilon \\ \varepsilon & \end{pmatrix}}_{\text{shear}} + \underbrace{\begin{pmatrix} & \varepsilon \\ -\varepsilon & \end{pmatrix}}_{\text{rotation}} = \underbrace{\begin{pmatrix} & 2\varepsilon \\ & \end{pmatrix}}_{\text{total}}$

$$\varepsilon_{12} = (\tau_{12} - \lambda \cdot 0)/2\mu \qquad G \equiv \tau \Big/ \frac{\Delta y}{\Delta x} = \frac{\tau}{2\varepsilon} = \frac{2\mu}{2} = \mu$$



Principal stress/strain in rotated frame, $\quad \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\sigma} \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{V} = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{V} \underbrace{\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}}_{D}$

$$\overleftrightarrow{\sigma}' = \begin{pmatrix} \tau & & \\ & -\tau & \\ & & 0 \end{pmatrix}_{45°} \quad \overleftrightarrow{\varepsilon}' = \begin{pmatrix} \varepsilon - \varepsilon' & & \\ & -\varepsilon + \varepsilon' & \\ & & \varepsilon' - \varepsilon' \end{pmatrix}$$

2 tensile stresses: $\varepsilon - \varepsilon' = \frac{\tau}{2\mu} \Rightarrow \boxed{G = \mu}$

In fluids shear strain is used to define viscosity $\eta = \tau / v_{y,x}$
(damping constant B instead of stiffness matrix K)

c) Bulk Modulus "K" : uniform pressure $\overleftrightarrow{\sigma} = \begin{pmatrix} -p & & \\ & -p & \\ & & -p \end{pmatrix}$

$V+\Delta V = (\ell+\Delta\ell)^3 \approx \ell^3 + 3\Delta\ell \qquad \frac{\Delta V}{V} = 3\,\frac{\Delta\ell}{\ell} = 3\,\varepsilon$

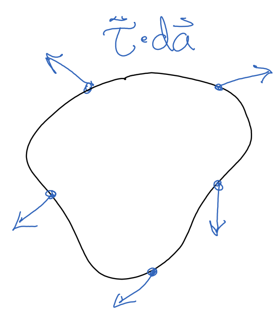
$\varepsilon = \varepsilon_{11} = \left(-p - \lambda\,\frac{-3p}{3\lambda+2\mu}\right)/2\mu = \frac{-p}{3\lambda+2\mu} \qquad K \equiv \frac{-P}{V/\Delta V}... = \boxed{\lambda+\frac{2}{3}\mu}$

d) P-wave "G" & S-wave "M" Modulus: see below: $\boxed{M=\lambda+2\mu}$

restoring force for longitudinal "pressure" & transverse "shear" waves

- Dynamics: Newton's law for bulk waves – extension of 1d and 2d waves

$$\int \nabla\cdot\overleftrightarrow{\tau}\,dV = \oint \overleftrightarrow{\tau}\cdot d\vec{a} = \Sigma F \qquad \nabla\cdot\overleftrightarrow{\tau} = \frac{\Sigma\vec{F} = m\vec{a}}{V} = \rho\ddot{\vec{u}}$$



use $\overleftrightarrow{\tau} = \lambda\,\mathrm{Tr}\,\overleftrightarrow{\varepsilon}\,\overleftrightarrow{I} - 2\mu\overleftrightarrow{\varepsilon}$ and $\varepsilon_{ij} = \frac{1}{2}(u_{i,j}+u_{j,i})$

to obtain the wave equation (with $\nabla^2 u$, etc)

- Vector derivatives for multidimensional waves

for 2d scalars, $\nabla_\perp^2\eta \equiv (\partial_x^2+\partial_y^2)\eta = \eta_{,ii}$

for 3d vectors, $\nabla^2\vec{u} \equiv (\nabla\cdot\nabla)\vec{u} = (\partial_x^2+\partial_y^2+\partial_z^2)\vec{u}$ or $(\nabla^2 u)_i = u_{i,jj}$

in analogy with $\hat{n}\cdot\hat{n} = \underbrace{\hat{n}\hat{n}\cdot}_{P_\parallel} - \underbrace{\hat{n}\times\hat{n}\times}_{P_\perp}$, we have $\nabla^2 = \nabla_\parallel^2 + \nabla_\perp^2$

where $\nabla_\parallel^2\vec{u} = \nabla\nabla\cdot\vec{u}$ or $(\nabla_\parallel^2\vec{u})_i = u_{j,ji}$
$\nabla_\perp^2\vec{u} = -\nabla\times\nabla\times\vec{u} = (\nabla^2-\nabla_\parallel^2)\vec{u}$

using these, one finds $M = \lambda+2\mu$ $G=\mu$ in 3d waves.

- References: https://en.wikipedia.org/wiki/Seismic_wave
  https://en.wikipedia.org/wiki/Linear_elasticity (uses σ, not τ for stress)
  https://en.wikipedia.org/wiki/Elastic_modulus (Lamé parameters, others)
  https://en.wikipedia.org/wiki/Viscosity (shear stress is used to define viscosity)