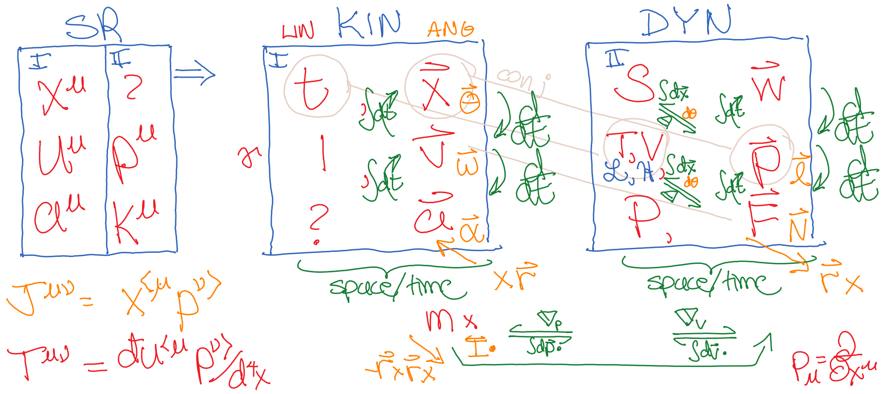
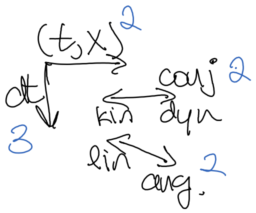
L27 Momentum

Wednesday, November 13, 2019 07:19

- Roadmap: finished oscillations (most important part of course)
  - Final topics of the course
    - i. central forces: 2 body problem - Kepler orbits, Rutherford scattering
    - ii. non-inertial frames: inertial forces - centrifugal, Coriolis
    - iii. rigid body: rotational motion - Euler angles/equations
  - A central theme is Degrees of Freedom (DOF)
    - The variables use to describe the state of the system
    - The Equation of Motion (EOM) describes evolution of DOF
      - NI,II on each particle, NIII on every pair, L,H on each DOF
    - Already talked about multiple particle DOF
      - Collective DOF (modes) simplify the physics -> more physical
    - Will formally treat multiparticle systems
      - 2 body: reduced mass equivalent to 1 body problem
      - Rigid bodies: 6 DOF = 3 trans + 3 rotations (Euler angles)
  - All 3 topics rely heavily on conservation principles
    - Already talked about conservation within Lagrangian framework
      - Review Lagrangian, Hamiltonian - conservation principles
      - Noether's theorem: symmetries <-> conserved currents
    - Will derive conservation principles straight from NII,III
      - i. Momentum ($\int \vec{F} dt$ NII, NIII)
      - ii. Angular Momentum ($\vec{r} \times$ everything)
      - iii. Energy ($\vec{F} \cdot d\vec{r}$ conservative forces)
    - Conserved quantities are "first integrals" - used to integrate the EOM
      - Eg. $\vec{F} = m\vec{a} = m\ddot{x}$ needs to be integrated twice to obtain x

        if $p = m\dot{x} = const$, then p only needs to be integrated once
  - Map of classical mechanics





- Conservation of momentum [Taylor 1.5]

$$p_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha} \qquad \dot{p}_\alpha = \frac{\partial \mathcal{L}}{\partial q^\alpha} \qquad \text{if } \mathcal{L} \text{ invariant w/r } q^\alpha \text{ then } \dot{p}_\alpha = 0$$

$$\text{NII}: \quad \vec{F} = m\vec{a} = m\frac{d}{dt}\vec{v} = \frac{d}{dt}m\vec{v} = \frac{d\vec{p}}{dt} \qquad d\vec{p} = \vec{F}dt$$

$$\text{NIII}: (\vec{F}_{12} = -\vec{F}_{21})\,dt \qquad d\vec{p}_1 = -d\vec{p}_2$$

(also true in special relativity!)

$$\frac{d}{dt}\left(\vec{P} \equiv \sum_i \vec{p}_i\right) = \sum_{i\neq j} \cancelto{\vec{F}_{ij}=-\vec{F}_{ji}}{F_{ij}^{int}} + \sum_i F_i^{ext} \equiv F^{ext} \qquad \text{if } \vec{F}_{ext}=0 \quad \vec{P} = \text{const}$$

- Centre of mass [Taylor 3.3]

$$\vec{p}_i = m\dot{\vec{r}}_i \qquad \text{what about } \vec{P} = M\dot{\vec{R}} \Rightarrow F^{ext} = M\ddot{\vec{R}} \quad ?$$

- 0th moment of mass: $\boxed{M \equiv \sum m_i}$ or $\int dm$

$$\text{then} \quad \int dt \left[\vec{P} = M\dot{\vec{R}} = \sum_i \left(\vec{p}_i = m_i\dot{\vec{r}}_i\right)\right]$$

- 1st moment of mass $\boxed{M\vec{R} \equiv \sum_i m_i\vec{r}_i}$ or $\int dm\,\vec{r}$ let $\alpha_i = \frac{m_i}{M}$

the moments of mass also 'play nice' with angular momentum and energy!

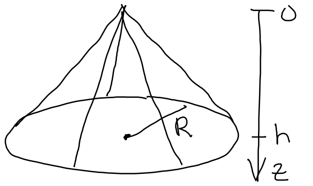
- Affine combination $\vec{R} = \alpha_1\vec{r}_1 + \alpha_2\vec{r}_2 + \dots$ where $\alpha_1 + \alpha_2 + \dots = 1$

on a space of points is the analog of the Linear combination on a linear space it is also called the weighted average, and is also used in other spaces, For example: statistical error:

$$W\bar{x} = \sum_i w_i x_i \quad \text{where} \quad w_i = (\delta_x^2)^{-1} \qquad W = \sum w_i$$

- Example: calculate the centre of mass of a cone of radius r and height h :

$$M = \int dm = \int \rho\pi r(z)^2 dz = \int \rho\pi\left(z\frac{R}{h}\right)^2 dz$$
$$= \rho\pi\frac{R^2}{h^2}\int_0^h z^2 dz = \rho\cdot\frac{1}{3}\pi R^2 h$$

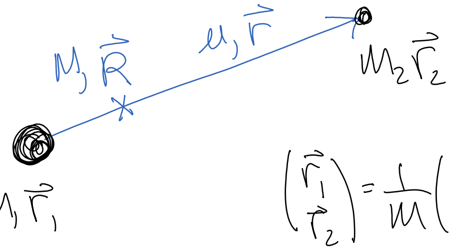


$$MZ = \int z\,dm = \rho\pi\frac{R^2}{h^2}\int_0^h z^3 dz = \rho\cdot\frac{1}{4}\pi R^2 h^2$$

$$\vec{R} = (0, 0, \tfrac{3}{4}h)$$

- Reduced mass:

$$\left(\begin{array}{l} M = m_1 \quad + m_2 \\ M\vec{R} = m_1\vec{r}_1 + m_2\vec{r}_2 \\ \vec{r} = -\vec{r}_1 + \quad r_2 \end{array}\right) \begin{array}{cc} \times 1 & \times 1 \\ \underline{\times m_2} & \underline{\times m_1} \\ - & + \end{array}$$



$$\begin{pmatrix} M\vec{R} \\ \vec{r} \end{pmatrix} = \begin{pmatrix} m_1 & m_2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \vec{r}_1 \\ r_2 \end{pmatrix}$$

$$\vec{r}_1 = \vec{R} - \frac{m_2}{M}\vec{r}$$
$$\vec{r}_2 = \vec{R} + \frac{m_1}{M}\vec{r}$$

$$\begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \end{pmatrix} = \frac{1}{M}\begin{pmatrix} 1 & -m_2 \\ 1 & m_1 \end{pmatrix}\begin{pmatrix} M\vec{R} \\ \vec{r} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{m_2}{M} \\ 1 & \frac{m_1}{M} \end{pmatrix}\begin{pmatrix} \vec{R} \\ \vec{r} \end{pmatrix}$$

$$\vec{P} = \vec{p}_1 + \vec{p}_2 = m_1\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2 = m_1\left(\dot{\vec{R}} - \frac{m_2}{M}\dot{\vec{r}}\right) + m_2\left(\dot{\vec{R}} + \frac{m_2}{M}\dot{\vec{r}}\right)$$
$$= (m_1+m_2)\dot{\vec{R}} + \frac{m_1 m_2}{m_1+m_2}(\dot{\vec{r}} - \dot{\vec{r}}) = M\dot{\vec{R}} + 0\,\mu\dot{\vec{r}}$$

$$\vec{L} = \vec{l}_1 + \vec{l}_2 = \vec{r}_1\times\vec{p}_1 + \vec{r}_2\times\vec{p}_2$$
$$= m_1\left(\vec{R} - \frac{m_2}{M}\vec{r}\right)\times\left(\dot{\vec{R}} - \frac{m_2}{M}\dot{\vec{r}}\right) + m_2\left(\vec{R} + \frac{m_1}{M}\vec{r}\right)\times\left(\dot{\vec{R}} + \frac{m_1}{M}\dot{\vec{r}}\right)$$
$$= \underbrace{(m_1+m_2)\vec{R}\times\dot{\vec{R}}}_{\text{sum of (first) products}} + \underbrace{\frac{m_1 m_2}{M}\vec{R}\times(\overset{0}{\cancel{\dot{\vec{r}} - \dot{\vec{r}}}})}_{\text{(outer)}} + \underbrace{\frac{m_1 m_2}{M}\vec{r}\times(\overset{0}{\cancel{-\dot{\vec{R}} + \dot{\vec{R}}}})}_{\text{(inner)}} + \underbrace{\frac{m_1 m_2^2 + m_2 m_1^2}{M^2}\vec{r}\times\dot{\vec{r}}}_{\text{(last)}}$$
$$= M\vec{R}\times\dot{\vec{R}} + \mu\vec{r}\times\dot{\vec{r}} = \vec{R}\times\vec{P} + \vec{r}\times\vec{p} \quad \text{[CM + relative "particles"]}$$

$$T = T_1 + T_2 = \tfrac{1}{2}m_1\dot{\vec{r}}_1^{\,2} + \tfrac{1}{2}m_2\dot{\vec{r}}_2^{\,2} = \tfrac{1}{2}m_1\left(\dot{\vec{R}} + \frac{m_2}{M}\dot{\vec{r}}\right)^2 + \tfrac{1}{2}m_2\left(\dot{\vec{R}} - \frac{m_1}{M}\dot{\vec{r}}\right)^2$$
$$= \underbrace{\tfrac{1}{2}(m_1+m_2)}_{\text{(first)}}\dot{\vec{R}}^2 + \frac{2}{2}\frac{m_1 m_2}{M}\underbrace{(\overset{0}{\cancel{2-2}})}_{\text{(cross)}}\dot{\vec{R}}\cdot\dot{\vec{r}} + \frac{1}{2}\frac{m_1 m_2^2 + m_2 m_1^2}{M^2}\dot{\vec{r}}^2$$
$$= \tfrac{1}{2}M\dot{\vec{R}}^2 + \tfrac{1}{2}\mu\dot{\vec{r}}^2 \quad \text{[CM + relative "particles"]}$$

Note: 1) p, L, T all separate into CM, relative co-ordinates!

2) relative co-ordinate has effective "reduced" mass $\mu = \frac{m_1 m_2}{m_1+m_2}$!