L28 Rotational Motion

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- *Conserved quantities are used to integrate the equations of motion*
  - *The central force 2-body problem (Kepler orbit) uses all three!*
  - *2 point-particles in 3d = 6 DOF x 2nd order* $\vec{F} = m\ddot{\vec{x}}$ *= 12 equations*
  - *Reduces to center of mass* $(M, \vec{R})$ *and reduced mass* $(\mu, \vec{r})$ *particles*
  - *CM also called center of momentum;* $\vec{P} = \sum m_i \dot{\vec{r}}_i = M\dot{\vec{R}}$ *is conserved*
  - *Reduced mass is 2-d (orbit in* $z = 0$ *plane) or 4 integrations* $\rho, \dot{\rho}, \phi, \dot{\phi}$
  - *First we eliminate* $\dot{\phi}$ *using conservation of angular momentum*
  - *Then we eliminate* $\dot{\rho}$ *using conservation of energy ->* $\rho'(\phi) = \cdots$

* Angular Motion $(\vec{r}\times)^n$!

- kinematics: arc length $\boxed{d\vec{s} = d\vec{\phi} \times \vec{r}}$ $\vec{\phi}$ NOT a vector! (doesn't add)

  angular velocity $\vec{\omega} = \dot{\vec{\phi}}$: $\vec{v} = \dfrac{d\vec{\phi}\times\vec{r}}{dt} = \dot{\vec{\phi}} \times \vec{r} = \vec{\omega}\times\vec{r}$

  angular acceleration $\vec{\alpha} = \dot{\vec{\omega}} = \ddot{\vec{\phi}}$: $\vec{a} = \frac{d}{dt}(\vec{\omega}\times\vec{r}) = \underbrace{\vec{\alpha}\times\vec{r}}_{\text{tangential}} + \underbrace{\vec{\omega}\times\vec{v}}$

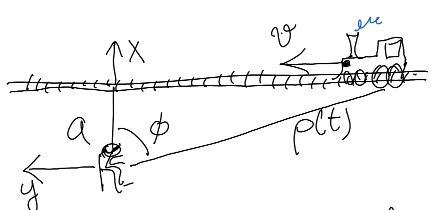
  $\vec{\omega}\times(\vec{\omega}\times\vec{r}) = -\dfrac{v^2}{r_\perp}\hat{r}_\perp$ centripetal

- dynamics: angular analog of $\vec{p}$: (constant $\vec{v}$)

  $x = a\tan\phi$

  $a\,m(v = a\sec^2\phi \cdot \dot{\phi})$

  $\vec{r}\times\vec{p} = \underbrace{m\rho^2}_{I}\underbrace{\dot{\phi}}_{\omega} = \ell$



- Lagrangian: $\ell = p_\phi = \dfrac{\partial}{\partial\dot{\phi}}\underbrace{\tfrac{1}{2}m(\rho\dot{\phi})^2}_{\frac{1}{2}I\omega^2} = \dfrac{\partial\mathcal{L}}{\partial\dot{\phi}}$ conserved if $N = \dfrac{\partial\mathcal{L}}{\partial\phi} = 0$.

- Rotational dynamics: $\dot{\vec{\ell}} = \vec{N}$ $\boxed{\vec{\ell} = \vec{r}\times\vec{p}}$

  $\vec{r}\times(\vec{F} - \dot{\vec{p}}) + (\dot{\vec{r}}\times\vec{p} = 0)$

  $\vec{r}\times\vec{F} = \frac{d}{dt}(\vec{r}\times\vec{p})$

  $\vec{N} = \dot{\vec{\ell}}$

  $\dot{\vec{\ell}} = \frac{d}{dt}(\vec{r}\times\vec{p}) = \cancel{\dot{\vec{r}}\times\vec{p}} + \vec{r}\times\dot{\vec{p}} = \underline{\vec{r}\times\vec{F} \equiv \vec{N}}$ or $\vec{\tau}$ or $\vec{\Gamma}$

- Inertia: $\vec{\ell} = \overleftrightarrow{I}\vec{\omega}$ $\overleftrightarrow{I} = m(r^2 I - \vec{r}\otimes\vec{r}) = m(r^T r\, I - r r^T)$

  $\vec{\ell} = \vec{r}\times\vec{p} = \vec{r}\times m(\vec{\omega}\times\vec{r}) = -m\vec{r}\times(\vec{r}\times\vec{\omega}) = m(r^2 - \vec{r}\,\vec{r}\cdot)\vec{\omega} \equiv \overleftrightarrow{I}\vec{\omega}$

  $\overleftrightarrow{I} = m(r^T r\, I - r r^T) = m\,(x\,y\,z)\begin{pmatrix}x\\y\\z\end{pmatrix} I - \begin{pmatrix}x\\y\\z\end{pmatrix}(x\,y\,z) = m\begin{pmatrix} y^2+z^2 & -xy & -xz \\ -yx & z^2+x^2 & -yz \\ -zx & -zy & x^2+y^2 \end{pmatrix}$

- Rotational Energy:

lin: $\vec{p} = m\vec{v}$ $\quad \vec{F} = \dot{\vec{p}} = m\vec{a}$ $\quad W = \int \vec{F}\cdot d\vec{x} = \int m\frac{d\vec{v}}{dt}\cdot d\vec{x} = \frac{1}{2}mv^2 = \frac{1}{2}\vec{p}\cdot\vec{v} = \frac{p^2}{2m}$

rot: $\vec{\ell} = \overleftrightarrow{I}\vec{\omega}$ $\quad \vec{N} = \dot{\vec{\ell}} = \overleftrightarrow{I}\vec{\alpha}$ $\quad W = \int \vec{N}\cdot d\vec{\theta} = \int \frac{d\vec{\omega}}{dt}\overleftrightarrow{I}\cdot d\vec{\theta} = \frac{1}{2}\vec{\omega}\cdot\overleftrightarrow{I}\cdot\vec{\omega} = \frac{1}{2}\vec{\ell}\cdot\vec{\omega} = \frac{1}{2}\vec{\ell}\cdot\overleftrightarrow{I}^{-1}\cdot\vec{\ell}$

- Summary: all formulas take on the same form, exchanging:

lin:	t	$\vec{r}\ \ \vec{v}\ \ \vec{a}$	m	$\vec{F}\ \ \vec{p}$	T, V
	$\Vert$	$\uparrow \times\vec{r}$	$\downarrow -\vec{r}\times\vec{r}\times$	$\downarrow \vec{r}\times$	$\Vert$
rot:	t	$d\vec{\theta}\ \ \vec{\omega}\ \ \vec{\alpha}$	$\overleftrightarrow{I}$	$\vec{N}\ \ \vec{\ell}$	T, V

See how angular variables fit on C.M. map, L31!

In particular, time and energy are the same for linear & rotational;
spatial coordinates: $d\vec{r} = d\vec{\theta}\times\vec{r}$ (arc length)
momentum, force : $\vec{\ell} = \vec{r}\times\vec{p}$ (to get $\rho^2\dot{\phi}$ – backwards!)
inertia: : $\overleftrightarrow{I} = -m\cdot\vec{r}\times\vec{r}\times$ (combining above two)

* Additional properties:

- $\vec{\ell}$ is conserved for any central Force $\vec{F}(\vec{r}) = F(r)\hat{r} = -\nabla V(r)$

$$\dot{\vec{\ell}} = \vec{N} = \vec{r}\times\vec{F} = \hat{r}r\times\hat{r}F(r) = \vec{0} \qquad \text{OR} \qquad \dot{\ell} = -\frac{\partial V}{\partial\phi} = 0$$

- KII the line from the sun to any planet sweeps out const. area/time: $\vec{\ell} = \vec{r}\times m\frac{d\vec{r}}{dt} = 2m\frac{d}{dt}\left(\frac{1}{2}\vec{r}\times d\vec{r} = da\right)$

- For any system of particles, the total $\vec{\ell} = \sum_i(\vec{\ell}_i = \vec{r}_i\times\vec{p})$ separates into $\vec{L} = \vec{R}\times\vec{P}$ of the C.M. + $\vec{\ell}' = \sum_i\vec{\ell}_i'$ relative to the CM

$$\vec{\ell} = \sum_i \vec{r}_i\times\vec{p}_i = \sum_i(\vec{R}+\vec{r}_i')\times m_i(\dot{\vec{R}}+\dot{\vec{r}}_i')$$

note: $\sum_i m_i\vec{r}_i' = \sum_i m_i(\vec{r}_i - \vec{R}) = M\vec{R} - M\vec{R} = \vec{0}$

$$= \sum_i \vec{R}\times m_i\dot{\vec{R}} + \cancel{\vec{R}\times m_i\dot{\vec{r}}_i'} + \cancel{\vec{r}_i'\times m_i\dot{\vec{R}}} + \sum\vec{r}_i'\times m_i\dot{\vec{r}}_i'$$

$$= \vec{R}\times M\dot{\vec{R}} + \sum_i(\vec{r}_i'\times m_i\dot{\vec{r}}_i' = \vec{\ell}_i') \equiv \vec{L} + \vec{\ell}'$$