L29 Energy

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- Conservation of energy is trickier, but most useful as a first integral
  - Impulse $\vec{F}dt = d\vec{p}$ transfers momentum from one particle to another
    - trivially applies to the free particle (CM of a system)
  - Angular momentum the same, but $\vec{r}\times$ everything; more useful for orbits
    - used to reduce 3-d problem to a single radial equation
  - Work $dW = \vec{F}\cdot d\vec{x} = dT = -dV$ transfers between `kinetic' and `potential' energy
    - conservation of energy $E = T + V = \frac{1}{2}m\dot{x}^2 + V(x)$ accounts for external forces
    - `potential momentum' $q\vec{A}$ also exists for external magnetic fields, not as common
    - E is also a first integral; second: $\frac{dx}{dt} = \sqrt{\frac{2}{m}(E - V(x))}$ or $\int dt = \int dx\left(\frac{2}{m}(E - V(x))\right)^{-\frac{1}{2}}$
- Historical debates between conserved energy `vis viva' (Leibniz, Maupertuis) vs. momentum `movientum' (Descartes, Newton) until we realized both were valid.
  - Culminated in thermodynamic laws, where scalar energy is more important, after Count Rumford observed heat produced while boring cannons. This elevated `vis viva' over the `caloric' theory (conservation of heat alone)
  - Vis viva was subsumed in the theory of `energy' coined by Thomas Young, 1807

1) Work-energy theorem $W = \Delta T$ (easy) - Coriolis, `quantité de travail mécanique'

$$dW \equiv \vec{F}\cdot d\vec{x} = m\vec{a}\cdot d\vec{x} = m\frac{d\vec{v}}{dt}\cdot d\vec{x} = m\vec{v}\cdot d\vec{v} = d\left(\tfrac{1}{2}mv^2\right) \equiv dT$$

2) Conservative force $W = -\Delta V$ (hard) relies on concepts from classical field theory
Essentially, if forces are symmetric in time with no short cuts or rabbit holes, then the work done against a force (potential) can be recovered as (kinetic) energy.

The Helmholtz theorem organizes the two complementary aspects of fields like $\vec{F}(\vec{r})$

Longitudinal / transverse separation of fields: $k^2\vec{v} = \underbrace{\vec{k}\vec{k}\cdot}_{P_\parallel}\vec{v} \underbrace{- \vec{k}\times\vec{k}\times}_{P_\perp}\vec{v}$

$$\nabla^2\vec{F} = \nabla\nabla\cdot\vec{F} + -\nabla\times\nabla\times\vec{F}$$

$$\boxed{\vec{F} = -\nabla\overbrace{\left(-\nabla^{-2}\underbrace{\nabla\cdot\vec{F}}_{\rho}\right)}^{V} + \nabla\times\overbrace{\left(-\nabla^{-2}\underbrace{\nabla\times\vec{F}}_{\vec{J}}\right)}^{\vec{A}}}$$

$$= -\nabla V + \nabla\times\vec{A}$$

potential $\downarrow d$
field
source $\downarrow S$ $-\nabla^2$

where $-\nabla^2 V = \nabla\cdot\vec{F} = \rho$ $\qquad -\nabla^2\vec{A} = \nabla\times\vec{F} = \vec{J}$

$$V = -\nabla^{-2}\rho = \int d\tau'\frac{\rho(r')}{4\pi\mathscr{r}} \qquad \vec{A} = \nabla^{-2}\vec{J} = \int d\tau'\frac{\vec{J}(\vec{r}')}{4\pi\mathscr{r}}$$

where $\vec{\mathscr{r}} = \vec{r} - \vec{r}'$

Helmholtz theorem: $\vec{F} = -\nabla V + \nabla\times\vec{A}$ is uniquely specified by its sources $\nabla\cdot F$, $\nabla\times\vec{F}$

We will treat the curl $\nabla\times\vec{F} = 0$ (conservation) in this lecture, and divergence $\nabla\cdot\vec{F} = \rho$ in the next.

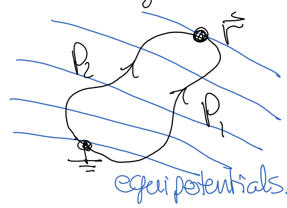
- *Irrotational* $\nabla\times\vec{F}=0$ *fields are conservative* $\vec{F}=-\nabla V$
  - The Fundamental Theorem of Calculus (FTC) states that sufficiently smooth functions have both derivatives and integrals, and that these operations are inverses of each other modulo a constant of integration. The situation in higher dimensions is a little more complicated: in general, $\omega = d\int\omega + \int d\,\omega$ so that the one-sided inverse exists only when the other term vanishes.
  - Conservative forces exploit two such theorems: the FTVC (vector calculus) and Stokes' theorem.

if $\nabla\times\vec{F}=0$ then $\exists\, V(\vec{r}) \ni \vec{F}=-\nabla V$ from above.

If there exists such a potential choose the point $\vec{r}_0 = \,\perp$ (ground) $\ni V(\vec{r}_0)=V_\perp=0$.

Then, by the FTVC, $V(\vec{r})-\cancel{V_\perp}^{\,0} = \Delta V = \int_\perp^{\vec{r}}\left(dV = \frac{\partial V}{\partial\vec{r}}\cdot d\vec{r} = \nabla V\cdot d\vec{r} = -\vec{F}\cdot d\vec{r}\right) \equiv -W$

For this integral to be well-defined, it should be path-independent

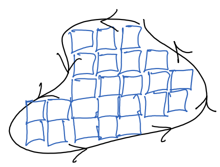


$$V(\vec{r}) = {}_{P_1}\!\int_\perp^{\vec{r}} -\vec{F}\cdot d\vec{r} = {}_{P_2}\!\int_\perp^{\vec{r}} -\vec{F}\cdot d\vec{r} \quad\text{or}\quad 0 = \oint_{\partial S = P_2 - P_1} -\vec{F}\cdot d\vec{r} = \int_S -\nabla\times\vec{F}\cdot d\vec{a}$$

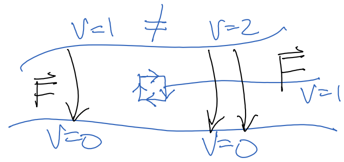
by Stokes' theorem. Since this must be true for any path, $\nabla\times\vec{F}=0$ where we started.

Geometrically, at each point, $\nabla\times\vec{F}=$ and neighboring sides cancel

so the integral over area = total circulation (integral around boundary)

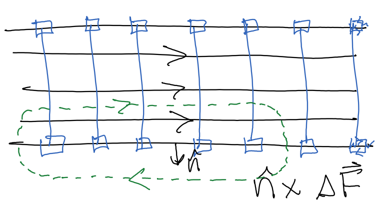


Curl or circulation equals ΔV between paths to the left vs. right, which prevents equipotentials from matching up.

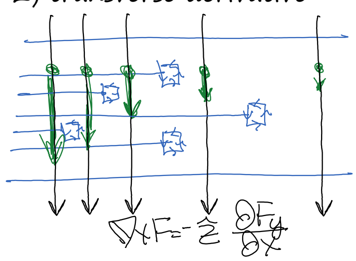


- Examples of non-conservative forces:

1) river bank



2) transverse derivative



3) eddy current