L30 - Inverse square law

Sunday, October 25, 2020 12:21

[Helmholtz]

$$\boxed{\vec F = -\nabla\big(\overbrace{-\nabla^{-2}\,\underbrace{\nabla\cdot\vec F}_{\rho}}^{V}\big) + \nabla\times\big(\overbrace{-\nabla^{-2}\,\underbrace{\nabla\times\vec F}_{\vec J}}^{\vec A}\big)}$$

$$= -\nabla V + \nabla\times\vec A$$

potential $\downarrow d$
field $\downarrow s$
source

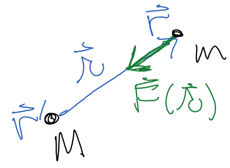
$-\nabla^{-2}$, ∇^2

Before using other theorems, let's review the point particle, inverse-square force.

$$\left[\; G \equiv \frac{1}{4\pi r} \;\underset{-\int d\ell\cdot}{\overset{-\nabla}{\rightleftarrows}}\; \frac{\hat r}{4\pi r^2} \;\underset{?}{\overset{\nabla\cdot}{\rightleftarrows}}\; \delta^3(\vec r) \;\right] \quad \times\, q/\varepsilon_0 \;\text{ OR }\; \times\, 4\pi G M m$$

$-\nabla^2$: $G \to \delta^3(\vec r)$

potential V — force $\vec F$ — source ρ



These derivatives/integrals invoke three Fundamental Thm's of Calculus:

The first two apply to any "conservative force" $F = -\nabla V$:

a) $-\nabla \frac{1}{4\pi r} = -\frac{\hat r}{1}\frac{\partial}{\partial r}\frac{1}{4\pi r} = \frac{\hat r}{4\pi r^2}$ or $-\int \frac{\hat r}{4\pi r}\cdot \hat r dr = \int \frac{-dr}{4\pi r^2} = \frac{1}{4\pi r}$

in general if $\vec F = -\nabla V$ then $W = \int \vec F\cdot d\vec r = \int -\nabla V\cdot d\vec r = -\int dV = -\Delta V$

b) $\nabla\times\frac{\hat r}{4\pi r} = \frac{1}{r^2 s_\theta}\begin{vmatrix} \hat r & r\hat\theta & r s_\theta \hat\phi \\ \partial_r & \partial_\theta & \partial_\phi \\ \frac{1}{4\pi r} & 0 & 0 \end{vmatrix} = 0.$ or $\nabla\times\vec F = 0$ (Fundamental Theorem of Vector Calculus)

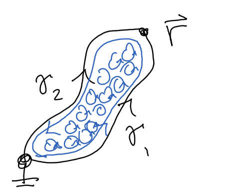
in general, if $\vec F = -\nabla V$ then $\nabla\times\vec F = -\nabla\times\nabla V = 0$

conversely, if $\nabla\times\vec F = 0$, then $W = -\oint \vec F\cdot d\vec\ell = -\int \nabla\times\vec F\cdot d\vec a = 0$

by Stoke's theorem; thus the work around a closed loop $= 0$.

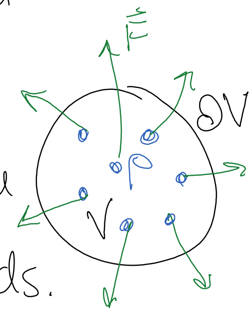
(work is conserved, $\vec F$ is a "conservative field")

$V \equiv -\int^{\vec r} \vec F\cdot d\vec r$ is well-defined since the difference



$$V_2 - V_1 = -\int_{\gamma_1}^{\vec r} \vec F\cdot d\vec r - -\int_{\gamma_2}^{\vec r} \vec F\cdot d\vec r = \oint_{\gamma_1-\gamma_2} \vec F\cdot d\vec r = 0$$

The third relates to the source of a conservative field:



c) $\nabla\cdot\frac{\hat r}{4\pi r^2} = \underbrace{\frac{1}{r^2}}_{\text{scale}}\frac{\partial}{\partial r}\underbrace{r^2}_{\text{factors.}}\cdot\frac{1}{4\pi r^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\frac{1}{4\pi} = \begin{cases}0 & \text{if } \vec r\neq 0\\ \infty & \text{if } \vec r = 0\end{cases}$

the scale factors account for the spreading out of area into space as the surface area of a sphere expands.

In general, $Q \equiv \int d\tau\,\rho(\vec r) = \int d\tau\,\nabla\cdot\vec F = \oint d\vec a\cdot\vec F \equiv \Phi$ [Gauss' law]

thus $\int d\tau\,\nabla\cdot\frac{\hat r}{4\pi r^2} = \int d\vec a\cdot\frac{\hat r}{4\pi r^2} = \int r^2 d\Omega\,\frac{1}{4\pi r^2} = \frac{1}{4\pi}\int_0^{\pi} dc_\theta\int_0^{2\pi} d\phi = \frac{4\pi}{4\pi} = 1$

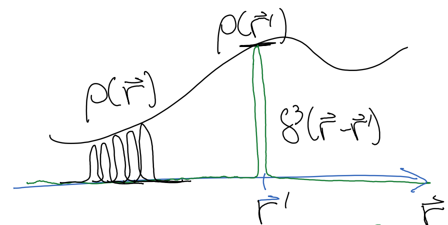
so $\nabla\cdot\frac{\hat r}{4\pi r^2} = \delta^3(\vec r)$ Inverse square law because const flux spreads out.

For $\vec F = \frac{-GMm\hat r}{r^2}$, Gauss' law states $\boxed{\oint \vec F\cdot d\vec a = -4\pi G M_{enc} m}$

Thus the inverse Laplacian of a point source $\rho(r) = \delta^3(\vec{\mathscr{r}})$

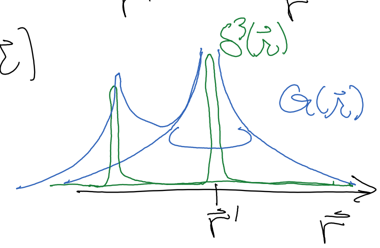
is $-\nabla^{-2}\delta^3(\vec r) = \frac{1}{4\pi\mathscr{r}} \equiv G(\mathscr{r})$ [definition of Green's function]

Treating any soure $\rho(\vec r) = \int d^3r'\,\delta^3(\vec r-\vec r')\rho(\vec r')$ as a "forest of poles" (sum of delta fns)



$$-\nabla^{-2}\rho(\vec r) = -\nabla^{-2}\int d\tau'\rho(\vec r')\,\delta^3(\vec r-\vec r') = \int d\tau'\rho(\vec r')\,(-\nabla^{-2})\,\delta^3(\vec{\mathscr{r}})$$

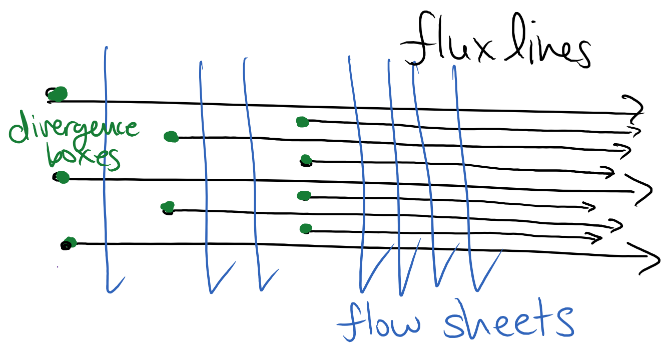
$$= \int d\tau'\rho(\vec r')\,G(\vec{\mathscr{r}}) = \int d\tau'\frac{\rho(\vec r')}{4\pi\mathscr{r}} = \int\frac{dq'}{4\pi\mathscr{r}}$$



• *Figures illustrating the geometry of vector fields*

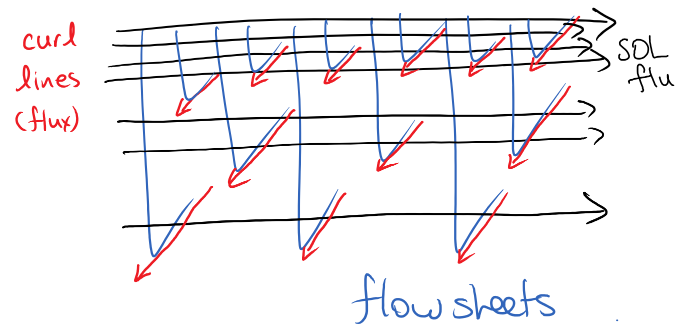
LONGITUDINAL

divergence: flux creation — $\mathcal{E}=0$ CONSERVATIVE $=-\nabla V$ — $\nabla\times=0$ IRROTATIONAL $\nabla\times=\vec{0}$

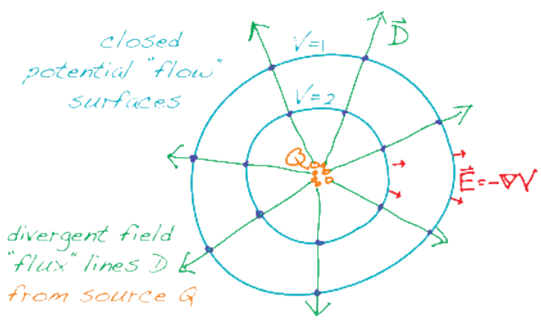


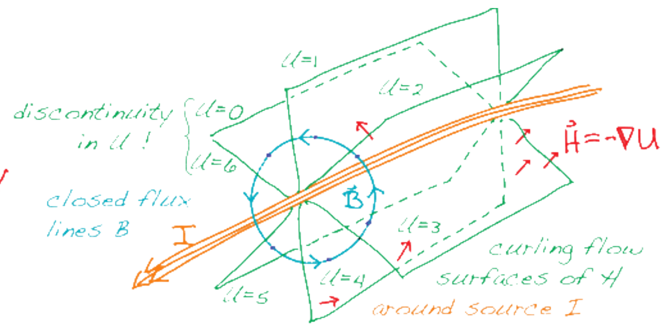
TRANSVERSE

curl: flow creation — $\Phi=0$ SOLENOIDAL $=\nabla\times\vec{A}$ — $\nabla\cdot=0$ INCOMPRESSIBLE $\nabla\cdot=0$



* generic fields have both components (types of sources)





$(W,\vec{p}) \xrightarrow{\rho} (P,\vec{F})$ Lorentz force

"force" (dynamic)

"source"

$$
\begin{tikzcd}
\chi \arrow[r, "d"] & (V,\vec{A}) \arrow[r, "d"] & (\vec{E},\vec{B}) \arrow[r, "d"] \arrow[d, shift right, "\varepsilon"'] \arrow[dr, "\sigma"] & 0 & \\
& (\vec{C},I) \arrow[r, "d"] & (\vec{D},\vec{H}) \arrow[r, "d"] \arrow[u, shift right, "\mu"'] & (\rho,\vec{J}) \arrow[r, "d"] & 0
\end{tikzcd}
$$

gauge — potentials — Maxwell — Continuity.

Noether's thm

EVERYTHING IS a derivative —OR— HAS a derivative.