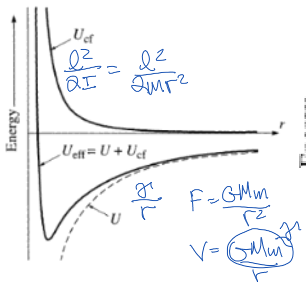
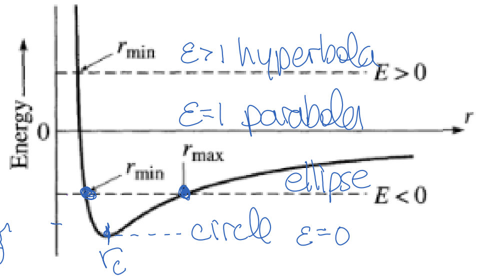
L31 Keplerian Orbit

Thursday, November 15, 2018 20:26





0) Circular orbit

$$F_c=\frac{mv^2}{r}=\frac{\gamma}{r^2}=F_g \qquad r_c=\frac{\gamma}{mv^2}=\frac{\ell^2}{\gamma m} \qquad \text{since } \ell=mvr$$

1) General method, H09 #2

$$\mathcal{H}=T+V=\tfrac{1}{2}m(\dot\rho^2+\rho^2\dot\phi^2)+V(r)=E$$
$$p_\phi=\frac{\partial T}{\partial\dot\phi}=m\rho^2\dot\phi=\ell$$

} "first integrals" constants of motion

$$\dot\rho^2=\frac{2}{m}\left(\varepsilon-\frac{\ell^2}{2m\rho^2}-V(\rho)\right)$$ "centrifugal potential"

$$dt=d\rho\left(\frac{2}{m}\left(E-\frac{\ell^2}{2m\rho^2}-V(\rho)\right)\right)^{-1/2} \qquad \frac{1}{\rho^2}=u^2$$ $\Leftarrow$ H09#2

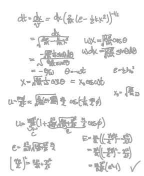
$$\frac{d}{dt}=\frac{d\phi}{dt}\frac{d}{d\phi}=\frac{\ell}{m\rho^2}\frac{d}{d\phi}$$ (trick #1) $dt=\frac{m\rho^2}{\ell}d\phi$

$$\frac{d\rho}{dt}=\frac{\ell}{m\rho^2}\frac{d\rho}{d\phi}=\frac{\ell}{m\rho^2}\frac{d}{d\phi}\left(\frac{1}{u}\right)=-\frac{\ell}{m}\frac{du}{d\phi}$$ (trick #2) $u=\frac{1}{\rho}$

$$d\phi=-\frac{\ell}{m}du\left(\frac{2}{m}\Big(E\underbrace{-\frac{\ell^2}{2m}u^2+\gamma u}_{\frac{\ell^2}{2m}\left(u-\frac{m\gamma}{\ell^2}\right)^2-\frac{m\gamma^2}{2\ell^2}}\Big)\right)^{-1/2} \qquad V(\rho)=-\frac{\gamma}{\rho}=-\gamma u$$

$$\frac{m}{\ell}d\phi=du\left(\frac{2}{m}\left(\underbrace{e+\frac{m\gamma^2}{2\ell^2}}_{\varepsilon}\underbrace{-\frac{\ell^2}{2m}}_{-\frac{1}{2}k}\underbrace{\left(u-\frac{m\gamma}{\ell^2}\right)^2}_{x^2}\right)\right)^{-1/2}$$

$$dt=\frac{dx}{v}=dx\left(\frac{2}{m}\left(\varepsilon-\tfrac{1}{2}kx^2\right)\right)^{-1/2}$$



Comparing with a spring, $u(\phi)$ is offset $\frac{m\gamma}{\ell^2}$, $\omega^2=\frac{k}{m}\to 1$

2) Special case: inverse-square-law

$$m\ddot\rho=F(\rho)+\frac{\ell^2}{m\rho^3} \qquad \text{let } u=\frac{1}{\rho} \qquad \dot\rho=-\frac{\ell}{m}u$$ do F=ma instead of conservation of energy

$$\ddot\rho=\frac{\ell}{mr^2}\frac{d}{d\phi}\left(\dot\rho=-\frac{\ell}{m}u'\right)=\frac{-\ell^2}{m^2r^2}u''=\frac{-\ell^2}{m^2}u^2u''$$ substitute in $\ddot\rho$ above:

$$u''=-u-\frac{m}{\ell^2u^2}F=-\left(u-\frac{m\gamma}{\ell^2}\right) \quad \text{for } F=-\gamma u^2$$

$$u-\frac{m\gamma}{\ell^2}=A\cos\theta \qquad \boxed{u=\frac{1}{c}(1+\varepsilon\cos\theta)} \qquad c=\frac{m\gamma}{\ell^2}$$ equation of an ellipse (conic section)