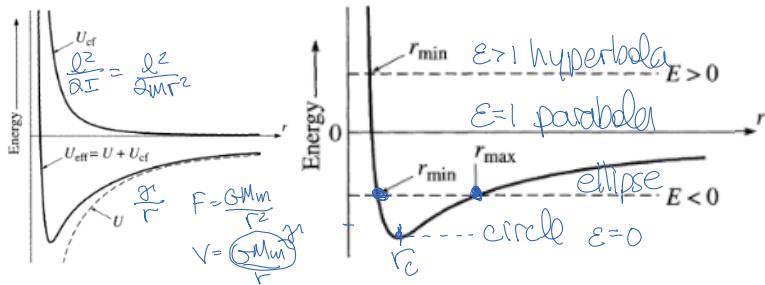


L31 Keplerian Orbit

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0) Circular orbit

$$F_c = \frac{mv^2}{r} = \frac{\gamma}{r^2} = F_g \quad r_c = \frac{l^2}{mr^2} = \frac{l^2}{\gamma m} \quad \text{since } l = mv^2$$

1) General method, HO9 #2

$$\mathcal{H} = T + V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + V(r) = E \quad \left. \begin{array}{l} \text{"first integrals"} \\ \text{constants of motion} \end{array} \right\}$$

$$p_\phi = \frac{d\phi}{dt} = m r^2 \dot{\phi} = l$$

$$\dot{r}^2 = \frac{2}{m}(E - \frac{l^2}{2mr^2} - V(r)) \quad \text{"centrifugal potential"}$$

$$dt = dr \left(\frac{2}{m} \left(E - \frac{l^2}{2mr^2} - V(r) \right) \right)^{-1/2} \quad \leftarrow \text{HO9#2}$$

$$\frac{dt}{dr} = \frac{dr}{dt} \frac{dt}{d\phi} = \frac{l}{mr^2} \frac{dt}{d\phi} \quad (\text{trick #1}) \quad dt = \frac{mr^2}{l} d\phi$$

$$\frac{d\phi}{dt} = \frac{l}{mr^2} \frac{dt}{dr} = \frac{l}{mr^2} \frac{dt}{d\phi} (t) = \frac{-l}{m} \frac{du}{d\phi} \quad (\text{trick #2}) \quad u = \frac{1}{r}$$

$$d\phi = -\frac{l}{m} du \left(\frac{2}{m} \left(E - \frac{l^2}{2mr^2} u^2 + \frac{1}{2}u \right) \right)^{-1/2} \quad V(r) = -\frac{\gamma}{r} = -\frac{1}{2}u^2$$

$$\frac{l^2}{2mr^2} (u - \frac{ml}{l^2})^2 - \frac{ml^2}{2r^2}$$

$$\frac{d\phi}{dt} = du \left(\frac{2}{m} \left(E + \frac{ml^2}{2r^2} - \frac{l^2}{2mr^2} (u - \frac{ml}{l^2})^2 \right) \right)^{-1/2}$$

$$dt = \frac{dx}{v} = dx \left(\frac{2}{m} \left(E - \frac{1}{2}kx^2 \right) \right)^{-1/2}$$

$$\begin{aligned} du + \frac{d\phi}{dt} dt &= du \left(\frac{2}{m} \left(E - \frac{1}{2}kx^2 \right) \right)^{-1/2} \\ &= \sqrt{\frac{2}{m}} \sqrt{E - \frac{1}{2}kx^2} dt \quad (E - \frac{1}{2}kx^2) \geq 0 \\ &\Rightarrow \sqrt{\frac{2}{m}} \sqrt{E - \frac{1}{2}kx^2} = \sqrt{E - \frac{1}{2}kx^2} dt \\ &\Rightarrow \sqrt{\frac{2}{m}} = \sqrt{E - \frac{1}{2}kx^2} dt \quad x \in \mathbb{R}_0 \\ &\Rightarrow \frac{2}{m} = \frac{E - \frac{1}{2}kx^2}{dt^2} \quad (E - \frac{1}{2}kx^2) \geq 0 \\ &\Rightarrow dt^2 = \frac{m(E - \frac{1}{2}kx^2)}{2} \quad E - \frac{1}{2}kx^2 \geq 0 \\ &\Rightarrow dt = \sqrt{\frac{m(E - \frac{1}{2}kx^2)}{2}} \quad E - \frac{1}{2}kx^2 \geq 0 \\ &\Rightarrow dt = \sqrt{\frac{m(E - \frac{1}{2}kx^2)}{2}} \quad E - \frac{1}{2}kx^2 \geq 0 \end{aligned}$$

Comparing with a spring, $u(\phi)$ is offset $\frac{ml}{l^2}$, $\omega^2 = \frac{k}{m} \rightarrow 1$

2) Special case: inverse-square-law

$$m\ddot{r} = F(r) + \frac{l^2}{mr^3} \quad \text{let } u = \frac{1}{r} \quad \dot{r} = -\frac{l}{m} u \quad \text{do } F = ma \text{ instead of conservation of energy}$$

$$\ddot{r} = \frac{l}{mr^2} \frac{d}{dt} \left(\frac{1}{r} \right) = \frac{-l^2}{m^2 r^2} u' = \frac{-l^2}{m^2} u^2 u'' \quad \text{substitute in } \ddot{r} \text{ above:}$$

$$u'' = -u - \frac{m}{l^2 u^2} F = -(u - \frac{ml}{l^2}) \quad \text{for } F = -\frac{1}{2}u^2$$

$$u - \frac{ml}{l^2} = A \cos \theta \quad \boxed{u = \frac{1}{C} (1 + e \cos \theta)} \quad C = \frac{ml^2}{l^2} \quad \text{eccentricity of the ellipse (conic section)}$$